First part: Group Actions — L28

Def. Given a top. space Y and a group G, an action of G on Y is a group homomorphism

$$\rho: G \longrightarrow \underbrace{\mathrm{Homeo}(Y)}_{\text{is a group w/ composition}}.$$

Example: $Y = \mathbb{R}^2$ w/ Euclidean top. $G = \mathbb{Z}^2$

$\rho((m,n))$ = translation of vector (m,n) in $\mathbb{R}^2$

group hom. because

$$\rho((m,n)) \circ \rho((m',n')) = \rho((m+m', n+n')).$$

We assume $\rho: G \to \mathrm{Homeo}(Y)$ is injective and write g instead of $\rho(g)$.

Basic Postulate: (properly discontinuous action)

(*) $\forall y \in Y \quad \exists\, U = U(y)$ neighb.

such that the collection $\{g(U)\}_{g \in G}$ consists of pairwise disjoint sets

$$\left[\, g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2 \,\right]$$

Comments:

1) Condition $(*)$ is certainly FALSE if the group action has fixed points i.e. $\exists\, y \in Y$, $g \in G$ w/ $g \neq$ identity and $g(y) = y$ (so there are lots of actions violating $(*)$ like e.g. groups of rotations in $\mathbb{R}^2$, around the origin: $\mathbb{Z}/_{n\mathbb{Z}}$ acting on $\mathbb{R}^2$, with generator the rotation of $2\pi/_n$ around origin).

2) if $p : \tilde{X} \to X$ is a covering space then $\underbrace{G(\tilde{X})}_{\text{deck transformations}}$ do satisfy $(*)$. Indeed, given $\tilde{x} \in \tilde{X}$ take an evenly covered neighb. U of $x = p(\tilde{x})$ and let $\tilde{U} \ni \tilde{x}$ w/

$$p|_{\tilde{U}} : \tilde{U} \to U \text{ homeomorphism.}$$

Say $\tau_1(\tilde{U}) \cap \tau_2(\tilde{U}) \neq \phi$, so $\exists\, \tilde{x}_1, \tilde{x}_2 \in U$ w/ $\boxed{\tau_1(\tilde{x}_1) = \tau_2(\tilde{x}_2)}$, hence

$$p\,\tau_1(\tilde{x}_1) = p\,\tau_2(\tilde{x}_2)$$

$$\begin{array}{ccc} \| & & \| \\ p(\tilde{x}_1) & & p(\tilde{x}_2) \end{array} \Rightarrow \tilde{x}_1 = \tilde{x}_2$$

$$\tau_1(\tilde{x}_1) = \tau_2(\tilde{x}_1) \iff \underbrace{\tau_2^{-1}\tau_1}(\tilde{x}_1) = \tilde{x}_1$$

$$\Rightarrow \tau_2^{-1}\tau_1 = id_{\tilde{X}} \Rightarrow \tau_1 = \tau_2 \qquad \square$$

What do we do w/ "good" group actions?

We take <u>quotients</u>!

In the setting above, consider the equivalence relation $\sim$ defined by

$$y_1 \sim y_2 \iff \exists g \in G \text{ w/ } y_1 = g(y_2)$$

i.e. we call <u>equivalent</u> points in the <u>same orbit</u>.

We denote by Y/G the quotient of Y modulo this relation $\sim$, w/ $p: Y \longrightarrow Y/G$ the projection.

quotient topology

<u>Example</u>: If $p: \tilde{X} \longrightarrow X$ is a <u>normal</u> cov. space then

$$\frac{\tilde{X}}{G(\tilde{X})} \underset{\text{homeo}}{\cong} X$$

i.e. we identify all points in any given fiber.

This fact is true in general!

Thm: If an action of a group G on a top. space Y satisfies $(*)$ then:

(a) the quotient map $p: Y \longrightarrow Y/G$ is a normal covering map

(b) if Y is path-connected, then G coincides w/ the group of deck transformations of this covering

(c) if Y is path-connected and locally path-connected then

$$G \underset{\text{group isomorphism}}{\cong} \frac{\pi_1(Y/G)}{p_*(\pi_1(Y))}.$$

Rmk. / Applications: if Y is simply-connected (e.g. $\mathbb{R}^n$ or S^n, $n \geq 2$) then statement (c) is in fact $\boxed{G \cong \pi_1(Y/G).}$

- $$\mathbb{R}^2/\mathbb{Z}^2 \underset{\text{homeo}}{\cong} S^1 \times S^1 \quad \text{(torus)}$$

hence thm. above gives $\pi_1(\mathbb{R}^2/\mathbb{Z}^2) \cong \mathbb{Z}^2$

So 3 proofs:
- i) product
- ii) Van Kampen
- iii) group action

Cf. Example 1.42, also for Klein bottle.

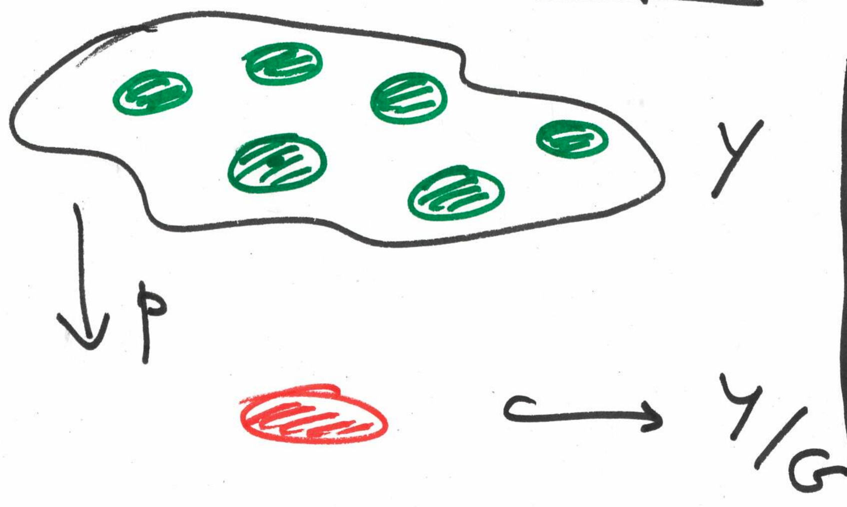
• $\underbrace{\mathbb{R}\mathbb{P}^n}_{\equiv \mathbb{P}^n(\mathbb{R})} \cong S^n\big/\mathbb{Z}_2$ $(n \geq 2)$

hence thm. above gives $\pi_1(\mathbb{R}\mathbb{P}^n) \cong \mathbb{Z}/2\mathbb{Z}$. (Cf. Example 1.43)

• This approach also allows to immediately determine the fundamental group of a large class of spaces, called LENS SPACES, obtained as quotients of $S^3 \subset \mathbb{C}^2 \cong \mathbb{C} \times \mathbb{C}$.

Proof.:

(a) $p : Y \longrightarrow Y/G$ is clearly continuous by def. of quotient topology, and surjective. Given $y \in Y$ let $U = U(y)$ satisfy axiom $(*)$, hence $p(U)$ is an evenly covered neigh. of $p(y)$, with preimage the disjoint family $\{g(U)\}_{g \in G}$



$p|_{g(U)} : g(U) \to p(U)$ bijection, and open

since $V \subset g(U)$

$p^{-1}(p(V)) = \coprod_{g' \in G} g'(V)$

□

<u>why normal?</u> we use the "<u>geometric condition</u>" (cf. L 26, L 27). Given $[y] \in Y/G$ points in its fiber have the form $\{g(y)\}_{g \in G}$ hence for fixed g a deck transformation taking y to $g(y)$ is simply g!

(b) trivially (by def. of group action)

$$G \subset \underbrace{G(Y)}_{\text{deck transformation}}$$

but in fact this is an <u>equality</u>, because if $\tau \in G(Y)$ one can just take $y \in Y$ and look at $\tau(y)$: have def (by def. of Y/G) $\tau(y) \sim y$ so can find $g \in G$ w/

$$\boxed{\tau(y) = g(y)}$$

hence from $\nearrow$ $g^{-1}\tau(y) = y$

$\xRightarrow[\text{w/ fixed point}]{\text{deck transformation}}$ $g = \tau \Rightarrow G = G(Y)$.

(c) Given part (b) just apply general theorem proven in L 27.

$\square$

Second part: topological approach to algebraic questions

So far we have employed Algebra to answer questions in Topology: for instance, is a sphere S^2 homeo to a torus T^2 ?

Answer: No, because if they were homeomorphic they would have the same fund. group (which is computable and different, w/ $\pi_1(S^2) = 0$, $\pi_1(T^2) = \mathbb{Z}^2$.)

One can also go the other way, i.e. interpret pure algebraic questions in geometric/topologic terms (Geometric group Theory).

Example of a question: is a subgroup of a free group also a free group? Hard to attack at the algebraic level! But one can rephrase it as a simple topological question.

Sketch (cf. Section 1.A)

- G ⟿ X top. space (in fact a combinatorial graph) having G as fundamental group

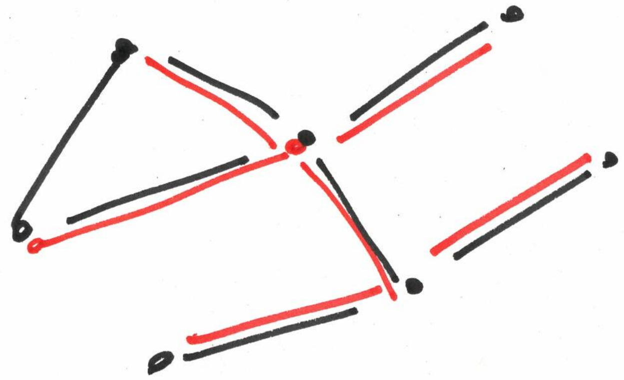
  ↑ free group

  (think of finitely generated case)

  E.g. if $G = \mathbb{Z} * \mathbb{Z}$ then $X = S^1 \vee S^1$.

- take $H < G$. By covering space theory $\exists\ \tilde{X}$ top. space (in fact a graph) such that $p: \tilde{X} \to X$ have $p_*(\pi_1(\tilde{X}, \tilde{x})) = H < G$.

- but now the fundamental group of any graph is always free and 'determined by # cycles'!



recipe: take a maximal tree, count the number of residual edges.

$\square$