Probability and Statistics

Exercise sheet 12

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1FuW9HQponei5ipS4j2J3lM4dfiP7MVQQ_0cMB1UUAXg/edit?usp=sharing

Exercise 12.1 Suppose that $X_1, \dots, X_n$ form a random sample from a Poisson distribution for which the mean λ is unknown. Determine the maximum likelihood estimator for λ.

Exercise 12.2 In the year 1910, Rutherford observed the radioactive decay of a substance in $n = 2608$ time intervals, each of 7.5 seconds. We use almost the same notation as Example 1.6.8 in the lecture notes: $\tilde{n}_k$ is the number of intervals with exactly k decays. We want to match a distribution to these data, and our null hypothesis H_0 is that the number of decays per interval is Poisson-distributed with unknown parameter λ.

Rutherford's experiments resulted in the following table:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	≥ 15
$\tilde{n}_k$	57	203	383	525	532	408	273	139	45	27	10	4	0	1	1	0

Table 1: Original table.

In order to fulfil the rule of thumb when a χ^2 asymptotic is an appropriate approximation, we merge the rare cases in the following way:

k	0	1	2	3	4	5	6	7	8	9	10	11	≥ 12
n_k	57	203	383	525	532	408	273	139	45	27	10	4	2

Table 2: Merged table.

(a) Do a χ^2 test with the given data. (You can use appropriate approximations.)

Hint: Remember what you have learned about χ^2 tests in the lecture. Use Exercise 12.1.

(b) Do a χ^2 test with the given data for the alternative null hypothesis H_0': The number of decays per interval is Poisson-distributed with (exogenously given) parameter $\lambda' = 3.87$.

(c) Do you think a Poisson distribution is a plausible model? Do you think H_0' is plausible?

Exercise 12.3 Let X be a normal random variable with $\mathbb{E}[X] = m$ and $\text{Var}[X] = \sigma^2 = 0.0014^2$. Let also X_i for $i = 1, \dots, n$ be i.i.d. random variables that share the same distribution with X. The following 12 realisations x_i of the random variables X_i were recorded:

1.00781 1.00646 1.00801 1.00833 1.00738 1.00687
1.00783 1.00936 1.00564 1.00543 1.00794 1.01060

(a) Perform a statistical test at a level of confidence $\alpha = 5\%$ for the null hypothesis $H_0 : \mu = 1.0085$ against the alternative hypothesis $H_A : \mu = 1.008$.

(b) Calculate the power of the test from part (a).

(c) What happens to the power calculated in part (b) when the alternative hypothesis is changed to $H'_A \ : \ \mu = 1.007$?

Exercise 12.4 In a study on the reliability of ball-bearings (in German: Kugellager), two samples of 10 pieces each of two different types of ball-bearings were tested. The number of rotations (in millions) until break-down were

type I	3.03	5.53	5.60	9.30	9.92	12.51	12.95	15.21	16.04	16.84
type II	3.19	4.26	4.47	4.53	4.67	4.69	12.78	6.79	9.37	12.75

Before the realization of this study, it was not clear which type was more reliable.

(a) Are we dealing with a paired sample? Please explain your answer.

(b) Perform a t-test for the null hypothesis "the expected number of rotations until break-down is the same for the two types of ball-bearings" with level 5%. (What are the model assumptions of a t-test?)

(c) Which other test would be a better alternative (fewer model assumptions and usually better power)? You can run that test in R.

Hint: (Clicking the following link will reveal the solution of (c).) You can find the R-code at `https://www.kaggle.com/jakobheiss/probstat2020-ex12-4/edit`.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.