Probability and Statistics

Exercise sheet 13

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1FuW9HQponei5ipS4j2J3lM4dfiP7MVQQ_0cMB1UUAXg/edit?usp=sharing

Exercise 13.1 In a new study on the reliability of ball-bearings (in German: Kugellager), two samples of two different types of ball-bearings were tested, one piece for each of two different types in one of 10 different scenarios. The resulting numbers of rotations until breakdown were

type I	3.03	5.53	5.60	9.30	9.92	12.51	12.95	15.21	16.04	16.84
type II	3.19	4.26	4.47	4.53	4.67	4.69	12.78	6.79	9.37	12.75

Each column represents one of the testing scenarios. Before the realization of this study, it was not clear which type was more reliable.

(a) Are we dealing with a paired sample? Please explain your answer.

(b) Perform a t-test for the null hypothesis "the expected number of rotations until break-down is the same for the two types of ball-bearings for each testing scenario" with level 5%. (What are the model assumptions of a t-test?)

(c) Which other test would be an alternative if you do not want to assume a normal distribution?

Hint: (Clicking the following link will reveal the solution of (c).) You can find the R-code at https://www.kaggle.com/jakobheiss/sol13-1/edit.

(d) Compare your results with Exercise 12.4 (the numbers in the table are the same), and discuss your conclusion.

Exercise 13.2 Consider the null hypothesis H_0: X has the density $f_0(x) = f(x)$ and the alternative H_A: X has the density X $f_1(x) = f(x-1)$ for the following cases:

(a) f is a standard normal density,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

(b) f is a Cauchy density,

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

Compute in both cases the form of the rejection region of the most powerful test (also known as the likelihood ratio test; see the Neyman-Pearson lemma). Which differences do you find.

Exercise 13.3 Consider $X_1, ..., X_n$ i.i.d. $\sim \mathrm{Exp}(\lambda)$, $\lambda \in \Theta = (0, +\infty)$. Recall that the density of $X_i \sim \mathrm{Exp}(\lambda)$ is given by $f_\lambda(x) = \lambda e^{-\lambda x} I_{(0,+\infty)}(x)$. We want to test $H_0 : \lambda = 1$ versus $H_A : \lambda = 2$.

(a) Apply the Neyman-Pearson lemma to find a most powerful test of level α based on $X = (X_1, ..., X_n)$.

Hint: We recall that if $Y_1, ..., Y_n$ are i.i.d. $\sim \mathrm{Exp}(\lambda_0)$, then $\sum_{i=1}^n Y_i \sim G(n, \lambda_0)$.

(b) What is the power of the Neyman-Pearson test you have found?

Hint: You can express your answer in terms of F_n and F_n^{-1}, the cdf and inverse cdf of a Gamma distribution with parameters n and 1, that we denote by $\mathrm{G}(n, 1)$.

(c) For $n = 10$, we observe the following sample:

1.009	0.132	0.384	0.360	0.206	0.588	0.872	0.398	0.339	1.079

What decision do you take if you want the level of the test to be equal to $\alpha = 0.05$? What about $\alpha = 0.01$?

Hint: The quantiles of the $G(10, 1)$ distribution of order 5% and 1% are 5.425 and 4.130, respectively.

Exercise 13.4 Again in the setup of Exercise 13.3, it turns out that the Neyman-Pearson test you found there in (a) is actually UMP at the level α for testing $H_0 : \lambda = 1$ versus $H'_A : \lambda > 1$. More precisely, the same NP test is the most powerful among all tests of level α for the alternative $H''_A : \lambda = \lambda_1$ for any $\lambda_1 \in \Theta'_A = (1, +\infty)$, not only for $\lambda \in \Theta_A = \{2\}$.

How can you see why this is true?

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.