Probability and Statistics

Exercise sheet 1

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1BN_IullEPWMdpFOStk4o-MiYOubm4bizttIce04x02c/edit?usp=sharing

Please take into class a smartphone, laptop or any other device capable of opening the webpage https://kahoot.it/.

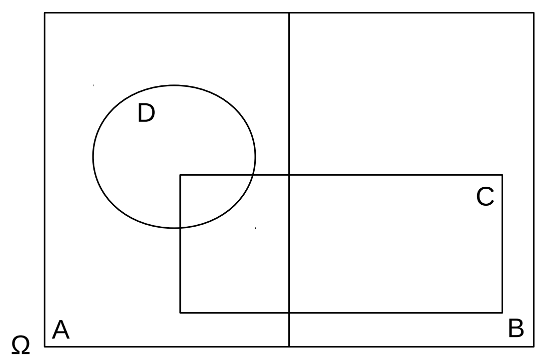
Exercise 1.1

(a) Let $A, B, C \subseteq \Omega$ be events. Which of the following notations make sense, and why or why not?

i) $\mathbb{P}[A \cup (B \cap C)]$
ii) $\mathbb{P}[A^c] \cap \mathbb{P}[B]$
iii) $\mathbb{P}[A] + \mathbb{P}[B]$
iv) $(\mathbb{P}[B])^c$

(b) For the following events, provide their graphic representations in the given Venn-diagram below.

i) $C \cap D$
ii) $(D \setminus C) \cup (C \cap A)$
iii) $B \cup D$



(c) We use an information channel to transmit four signals. Every signal is either transmitted correctly or incorrectly. We choose Ω to be the space of all 0-1 sequences of length four:

$$\Omega = \{\omega = (x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \{0, 1\}\},$$

i.e. $\Omega = \{(0,0,0,0), (1,0,0,0), (0,1,0,0), \dots, (1,1,1,1)\}$ and we interpret $x_i = 1$ as "signal i transmitted correctly" and $x_i = 0$ as "signal i transmitted incorrectly" for $i = 1, \dots, 4$. Further, we consider the following events:

- A : "Exactly one signal was transmitted incorrectly"
- B : "At least two signal were transmitted correctly"
- C : "At most two signal were transmitted correctly".

i) Write the events A, B and C as subsets of Ω.

ii) Describe the events $B \cap C$, $A \cup B$ and $A^c \cap C^c$ in words.

Exercise 1.2 Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Using induction, prove that for any events $A_1, ..., A_n \in \mathcal{F}$, we have the inclusion-exclusion formula

$$\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbb{P}\left[\bigcap_{j=1}^{k} A_{i_j}\right].$$

(This formula (1.7) in the lecture notes holds true for a general probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and even a bit more generally.) You may, assume that Ω is discrete and $\mathcal{F} = 2^{\Omega}$ to remain in the framework of the course so far.

Exercise 1.3

(a) Alice rolls a die and pays the square of the resulting number to Bob in CHF.

i) Define a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that describes rolling the die.

ii) Define a random variable X that describes how many CHF Bob receives, and write down its distribution.

iii) Calculate the expected value $\mathbb{E}[X]$.

(b) Three people each toss a fair coin. What is the probability of someone being the "odd man out"? This means that two of the players obtain the same outcome, while the third gets a different one. Please start solving this problem by defining a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Exercise 1.4

(a) Construct a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} = 2^{\Omega}$ and discrete random variables $X, Y : \Omega \to \mathbb{R}$, such that

i) $\mathbb{P}[X < \infty] = 1$ and $\mathbb{E}[X] = \infty$.

ii) $\mathbb{E}[Y] < \infty$ and $\mathbb{E}\left[Y^2\right] = \infty$.

(b) Is it possible to construct on a discrete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a discrete random variable $Z : \Omega \to \mathbb{R}$ such that $\mathbb{E}[Z] = \infty$ and $\mathbb{E}\left[Z^2\right] < \infty$?

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.