Probability and Statistics

Exercise sheet 4

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1S3l65UP2RJXyckisID_Q4WiinUeMF83HS_CMorD67Hg/edit?usp=sharing

Exercise 4.1 We have three dice. Two are ordinary with the numbers $1, 2, 3, 4, 5, 6$ and one is special where 6 is replaced by 7 (i.e. $1, 2, 3, 4, 5, 7$).

We first roll a ordinary die to decide which of the other two dice is chosen afterwards. If rolling the first die results in a number less than or equal to 4, we choose the second ordinary die, otherwise we choose the special die.

Then we roll the chosen die and denote its result by X_2.

(a) What is the distribution of X_2? Construct a minimal probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that one can answer this question and question (b).

(b) What is the conditional probability $\mathbb{P}\left[\text{first die} \geq 5 | X_2 = 5\right]$?

(c) Let X_1 be the result of the first die. We want to find the distribution of $X_1 + X_2$. Can we achieve this on the above defined probability space $(\Omega, \mathcal{F}, \mathbb{P})$? If not, construct a suitable probability space and find the distribution.

Exercise 4.2 (Simpson's paradox).

We are interested in studying the probability of success of a student at an entrance exam for two departments of a university. Consider the following events:

$$A := \{\text{The student is a man}\}$$
$$A^c = \{\text{The student is a woman}\}$$
$$B := \{\text{The student applied for department I}\}$$
$$B^c = \{\text{The student applied for department II}\}$$
$$C := \{\text{The student was accepted}\}$$
$$C^c = \{\text{The student was not accepted}\}$$

We assume the following probabilities:

$$\mathbb{P}\left[A\right] = 0.73,$$
$$\mathbb{P}\left[B \mid A\right] = 0.69, \mathbb{P}\left[B \mid A^c\right] = 0.24,$$
$$\mathbb{P}\left[C \mid A \cap B\right] = 0.62, \mathbb{P}\left[C \mid A^c \cap B\right] = 0.82,$$
$$\mathbb{P}\left[C \mid A \cap B^c\right] = 0.06, \mathbb{P}\left[C \mid A^c \cap B^c\right] = 0.07.$$

(a) Draw a tree describing the situation with the probabilities associated.

(b) From examining the probabilities in the tree, do you think that women are disadvantaged in the selection process? Why or why not?

(c) Calculate $\mathbb{P}[C \mid A]$ and $\mathbb{P}[C \mid A^c]$, i.e., the acceptance probabilities for men and women. Does this agree with your answer in (b)? Can you explain what is going on?

Exercise 4.3 (Monty Hall problem). You are on a game show, and you are given the choice of three doors. Behind one door is a car, behind the others are goats. You pick a door and the host, who knows what is behind the doors, opens another, behind which is a goat. He then asks you, "Do you want to keep your initial chosen door or do you want to switch to the other one?". Assuming that you like cars but not goats, what should you do?

(a) Construct a suitable model where you can answer this question with the help of conditional probabilities.

(b) Try to find an alternative solution (which of course must give the same answer).

Exercise 4.4 Let $(S_n)_{n=0,1,\ldots,N}$ be a random walk and T_0 the time of its first return to 0. Prove in detail that

$$\mathbb{P}[T_0 > 2n \mid X_1 = +1] = \mathbb{P}[T_{-1} > 2n - 1],$$

if $2n < N$ holds.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.