Probability and Statistics

Exercise sheet 5

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/17kYEAiK7wyR1ACDIEjVhheBRvT9ieLp8i_9qTWguI5A/edit?usp=sharing

Exercise 5.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a discrete probability space and $\mathcal{G} = (B_i)_{i\in I}$ a countable partition of Ω. A discrete random variable X is called $\mathcal{G}$-*measurable* if it can be written as $X = \sum_{i\in I} c_i I_{B_i}$ with (not necessarily distinct) $c_i \in \mathbb{R}$.

Two set systems $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ are called *independent* if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. A discrete random variable Y is called *independent* of $\mathcal{A}$ if the systems $\mathcal{B} := \{\{Y = c\} : c \in \mathbb{R}\}$ and $\mathcal{A}$ are independent.

(a) Suppose that X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$. Show that for any $F : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\mathbb{E}[F(X,Y) \mid \mathcal{G}](\omega) = \mathbb{E}[F(x,Y)]\big|_{x=X(\omega)}, \tag{1}$$

provided that all expectations are well defined.

(b) Deduce that $\mathbb{E}[X \mid \mathcal{G}] = X$ and $\mathbb{E}[Y \mid \mathcal{G}] = \mathbb{E}[Y]$

Exercise 5.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a discrete probability space, $\mathcal{G} = (B_i)_{i\in I}$ a countable partition of Ω and $(X_k)_{k\in\{1,\dots,n\}}$ a family of discrete random variables. Let the collection of $(\mathcal{G}, X_1, X_2, \dots, X_n)$ be independent, i.e. the events $(B_i, \{X_1 = x_1\}, \{X_2 = x_2\}, \dots, \{X_n = x_n\})$ are independent for all $B_i \in \mathcal{G}$ and $x_k \in X_k(\Omega)$. Show that $Y : \Omega \to \mathbb{R}, \omega \mapsto f(X_1(\omega), \dots, X_n(\omega))$ is independent of $\mathcal{G}$ for every $f : \mathbb{R}^n \to \mathbb{R}$.

Exercise 5.3 Let $(S_n)_{n=0,1,\dots,N}$ be a random walk and recall the family $\mathcal{F}_n$ of events observable up to time n. Every $A \in \mathcal{F}_n$ can be written as a union of sets from a partition $\mathcal{G}_n$ of Ω, and we define

$$\mathbb{E}[Z \mid \mathcal{F}_n] := \mathbb{E}[Z \mid \mathcal{G}_n]$$

for any random variable Z. Let $Y_n = \exp(S_n/\sqrt{N})$ for $n = 0, 1, \dots, N$ and define $Z_n := \mathbb{E}[Y_N \mid \mathcal{F}_n]$ for $n = 0, 1, \dots, N$.

(a) Show that $Z_n := Y_n \left(\cosh(1/\sqrt{N})\right)^{N-n}$ for $n = 0, 1, \dots, N$.

(b) Prove that $\mathbb{E}[Z_n \mid \mathcal{F}_m] = Z_m$ for $m \le n$. (This means that Z is a martingale.)

Hint: Show first that $S_N - S_n$ is independent of $\mathcal{G}_n$.

Exercise 5.4 In a clinical trial with two treatment groups, the probability of success in one treatment group is 0.5, and the probability of success in the other is 0.6. Suppose that there are five patients in each group. Assume that the outcomes of all patients are independent.

(a) Calculate the probability that the first group will have at least as many successes as the second group.

(b) Write the solution in a general formula, when both groups have size n ans success probabilities p_1 respectively p_2.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.