Probability and Statistics

Exercise sheet 6

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1H1zoIGZnkVz2t_9ZUboDqsvveflnZLXnWibmk8QJc_I/edit?usp=sharing

Exercise 6.1 (Intuition for σ-algebras) We throw simultaneously two dice, one green and one red. Consider the following events:

- $W_1 :=$ Neither of the dice has a result strictly greater than 2.
- $W_2 :=$ The green and the red one show the same number.
- $W_3 :=$ The number on the green die is 3 times the number on the red.
- $W_4 :=$ The number on the red die is greater by one than the number on the green die.
- $W_5 :=$ The number of the green die is not equal to the number on the red die.

(a) Write down a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where all of these events can be defined.

(b) Describe the W_i as subsets of Ω.

(c) Suppose you are colorblind (you cannot differentiate green and red).

(i) Write down an alternative σ-algebra $\tilde{\mathcal{F}}$ that captures the information available to a colorblind observer.

(ii) Which W_i is measurable with respect to $\tilde{\mathcal{F}}$, i.e., which W_i are in $\tilde{\mathcal{F}}$?

(iii) Define Y to be the result of the red die. Calculate $\mathbb{E}\left[Y \mid \tilde{\mathcal{F}}\right]$.

Exercise 6.2

(a) Let $(\mathcal{A}_i)_{i\in I}$ be a family of σ-algebras, where $I \neq \emptyset$ is and arbitrary index set. Show that $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra.

(b) Prove that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras, then $\mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-algebra if and only if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ or $\mathcal{A}_2 \subseteq \mathcal{A}_1$.

(c) Let $\mathcal{F}_0 \subseteq 2^\Omega$ be any system of subsets. Show that

$$\mathcal{F} = \sigma(\mathcal{F}_0) := \bigcap_{\substack{\mathcal{B}\supseteq\mathcal{F}_0 \\ \mathcal{B} \text{ is } \sigma\text{-algebra}}} \mathcal{B} \qquad \text{(lecture notes 1.6)}$$

is a σ-algebra.

Exercise 6.3 Recall that for a given $0 < \alpha < 1$, we defined the α-*quantile* of X (or of F) as

$$q_\alpha := F^{-1}(\alpha) := \inf\{x \in \mathbb{R} : F(x) \geq \alpha\},$$

where F is a given cumulative distribution function (cdf) of X.
Remark: Recall that the function F^{-1} is also called the (generalized) inverse of the cdf F. If $\alpha = \frac{1}{2}$, $q_{\frac{1}{2}}$ is called the *median*.

(a) Show in detail that F^{-1} is non-decreasing on $(0,1)$.

(b) Toss a fair coin n times and record

$$X_i = \begin{cases} 1 & \text{if head at the } i\text{th toss} \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$Y_n = \sum_{i=1}^{n} X_i$$

be the number of heads obtained in the n tosses.

Find the cdf of Y_n. Call it F_n.

(c) What is the median of F_n when n is even and when n is odd?

Exercise 6.4 We define the *variance* of a random variable X as

$$\mathrm{Var}\left[X\right] := \begin{cases} \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}\left[X\right]^2, & \mathbb{E}\left[|X|\right] < \infty, \\ \infty, & \text{else.} \end{cases}$$

(a) Suppose that X is a random variable. Show that $\mathbb{E}\left[X^2\right] < \infty$ if and only if $\mathrm{Var}\left[X\right] < \infty$.

(b) Suppose that X is discrete and $\mathrm{Var}\left[X\right] < \infty$. Show that $\mathbb{E}\left[X\right]$ minimizes the function

$$a \mapsto E[(X-a)^2] \quad (a \in \mathbb{R}), \tag{1}$$

with the help of Satz 5.1 in the lecture notes.

(c) Show the same statement as in (b), but this time without assuming that X is discrete and without the help of Satz 5.1.

(d) Show that the median $F^{-1}(0.5)$ of X minimizes the function.

$$a \mapsto \mathbb{E}\left[|X-a|\right] \quad (a \in \mathbb{R}). \tag{2}$$

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.