Probability and Statistics

Exercise sheet 7

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1Ut_tJaAxowFDeiHU8ZAYRPn29doXL8DFG4M9rPjJKA8/edit?usp=sharing

Exercise 7.1 Let $X_1, \dots, X_n$ be i.i.d. with distribution function (cdf) F.

(a) Let $S_n := \max_{1 \le i \le n} X_i$. Find the cdf of S_n as a function of F.

(b) Do the same for $I_n := \min_{1 \le i \le n} X_i$.

(c) Fix $x \in \mathbb{R}$ such that $F(x) \in (0,1)$. What is the limit of the cdf of S_n at x as $n \to \infty$? What about the cdf of I_n? How would you interpret these results? What does this mean if $X_1, \dots, X_n$ take values in a finite set $\{\xi_1, \dots, \xi_k\}$? To analyse the last question, compute $\mathbb{P}[|S_n - \xi_k| > \delta]$ and $\mathbb{P}[|I_n - \xi_1| > \delta]$ for $\delta > 0$.

Exercise 7.2

(a) Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ with

$$\sum_{n\in\mathbb{N}} \mathbb{P}[A_n] = \infty$$

and $\mathbb{P}\left[\bigcap_{n\in\mathbb{N}} \bigcup_{k\ge n} A_k\right] = 0$.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Take a sequence $(U_n)_{n\in\mathbb{N}}$ of i.i.d. random variables with uniform distribution $\mathcal{U}(0,1)$.

(i) Show that

$$\mathbb{P}\left[\exists \alpha > 1 : \liminf_{n\to\infty} n^\alpha U_n \in \mathbb{R}\right] = 0.$$

Hint: It may be useful to define $A_n^\alpha := \{U_n < n^{-\alpha}\}$ for $\alpha > 1$. Remember that the countable union of sets of probability 0 has probability 0.

(ii) Prove that

$$\mathbb{P}\left[\liminf_{n\to\infty} nU_n \in \mathbb{R}\right] > 0.$$

Exercise 7.3

(a) Let $X \sim \mathcal{U}(0,1)$. Compute $\mathbb{E}[X^n]$, $\mathbb{E}\left[X^{\frac{1}{n}}\right]$ for $n \in \mathbb{N}$, and $\Psi_X(t) := \mathbb{E}\left[e^{tX}\right]$ whenever these are defined. For which t is this the case?

(b) Let $X \sim \mathrm{Exp}(\alpha)$ for $\alpha > 0$. Derive the cdf of X and $\mathbb{E}[X^n]$ for $n \ge 1$ from the density function.

Remark: Watch out for the parametrisation—different sources use different parametrisations. In the lecture we consider the density function of an $\mathrm{Exp}(\alpha)$-distributed random variable to be $f(x) = \alpha e^{-\alpha x}$ for $x \ge 0$.

Exercise 7.4 An auto towing company services a 50 mile stretch of a highway. The company is located 20 miles from one end of the stretch, but inside the stretch. Breakdowns occur uniformly along the highway, and the towing trucks travel at a constant speed of 50mph. Find the mean and variance of the time elapsed between the instant the company is called and the instant a towing truck arrives at the breakdown.

Where is the optimal location for the company if they want to minimize the expected waiting time?

The optimal location is the median of X, because of Exercise 6.4(d). Since we assumed that $X \sim \mathcal{U}(0, 50)$, the median of X is 25. So the optimal location is in the middle of the stretch, as one expects from intuition or symmetry.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.