Probability and Statistics

Exercise sheet 8

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1hi3j55Wg1huC48lQXi81KPiKGEyaeBuMZXyXI8sx2Ro/edit?usp=sharing

Exercise 8.1

(a) Let

$$f(x) := \frac{1}{x^k} I_{[1,+\infty)}(x).$$

For which values of k, if any, is f a density function? Whatwould change if we consider instead $g(x) := cf(x)$ with $c > 0$?

(b) Give an example of a density function f such that $c\sqrt{f}$ cannot be a density function for any $c > 0$.

(c) Let

$$f(x) = c\,|x|\,(1 - x^2) I_{[-1,1]}(x).$$

(i) Find $c > 0$ such that f is a density function.

(ii) Find the cdf corresponding to this density.

(iii) Compute $\mathbb{P}\left[X < -\frac{1}{2}\right]$ and $\mathbb{P}\left[|X| \leq \frac{1}{2}\right]$.

Exercise 8.2

(a) Consider a random variable $X \sim \mathcal{U}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Find $\mathbb{E}[\sin X]$ and $\mathrm{Var}[\sin X]$.
Hint: $\sin(x)\sin(y) = 1/2(\cos(x-y) - \cos(x+y))$

(b) The lengths of the sides of a triangle are X, $2X$ and $2.5X$ with $X \sim \mathcal{U}(0, \alpha)$ for some $\alpha > 0$.

- Find the mean and variance of its area.
  Hint: Recall that if
  $$s = \frac{a+b+c}{2}$$
  with a, b, c the lengths of the sides, then the area of the triangle is
  $$|\Delta| = \sqrt{s(s-a)(s-b)(s-c)}$$
  (Heron's formula).
- How should we choose α so that the mean area is ≥ 1?

Exercise 8.3 It costs 1 dollar to play a certain slot machine in Las Vegas. The machine is set by the house to pay 2 dollars with probability 0.45 and nothing with probability 0.55.

Let X_i be the house's net winnings on the i^{th} play of the machine.

Let $S_n := \sum_{i=1}^{n} X_i$ be the house's winnings after n plays of the machine. Assuming that successive plays are independent, find:

(a) $\mathbb{E}[S_n]$;

(b) $\mathrm{Var}[S_n]$;

(c) the approximate probability that after $10,000$ rounds of the machine, the house's winnings are between 800 and $1,100$ dollars.

Exercise 8.4 Consider the joint density

$$f_{X,Y}(x,y) = \begin{cases} cxy, & 1 \le x \le 3 \text{ and } 1 \le y \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the normalising constant c.

(b) Are X and Y independent? Why?

(c) Find $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[XY]$.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.