Probability and Statistics

Exercise sheet 9

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1K14RnGQ-kDbbGHLlMgbp26nJ6EvnJRu3T4ufl_Pax_8/edit?usp=sharing

Exercise 9.1 Let $X_1, X_2, \dots, X_n$ be i.i.d. Cauchy-distributed, i.e. with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. Show that $\bar{X}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$ is again Cauchy-distributed using two different approaches:

(a) Use the convolution formula (2.2.4) to prove the claim.

Hint: Calculating the integral that appears in this proof would require a tedious partial fraction decomposition or the use of the residue theorem. You can skip these calculations by using

$$\int_{-\infty}^{+\infty} \frac{1}{(a^2+y^2)(b^2+(x-y)^2)} dy = \frac{\pi(a+b)}{ab\left((a+b)^2+x^2\right)}.$$

(b) Use characteristic functions to prove the claim.

(c) How does this fit together with the weak law of large numbers, the strong law of large numbers and the central limit theorem?

Exercise 9.2 We should like to compute

$$A := \int_{-3}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

by using the Monte Carlo method:

(a) Express A in the form $\mathbb{E}[f(X)]$, where X is a standard normal random variable and f an appropriate function.

(b) Take an i.i.d. family $(X_i)_{i\in\mathbb{N}}$ having the same distribution as X. Set

$$A_n := \frac{1}{n}\sum_{i=1}^{n} f(X_i).$$

What is the distribution of $A_n - A$?

(c) Compute $\mathbb{E}[A_n]$ and show that $\mathrm{Var}[A_n] = (A - A^2)/n$.

(d) Show that for any $x > 0$, $\mathbb{P}[|A_n - A| \geq x] \leq 1/(4nx^2)$. This means that $A_n - A$ converges to 0 in probability when $n \to \infty$.

(e) Which theorem can you apply to get directly the above convergence?

(f) Does $A_n - A$ converge to 0 almost surely too?

Exercise 9.3 Compute $\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n} \frac{n^k}{k!}$.
Hint: You can use the central limit theorem for i.d.d. random variables $(X_i)_{i\in\mathbb{N}}$ such that $X_i \sim \mathrm{Poisson}(1)$.

Exercise 9.4 Let X and Y be two independent standard normal random variables. Define the random variable

$$Z := X \operatorname{sign}(Y) := \begin{cases} X & \text{if } Y \geq 0, \\ -X & \text{if } Y < 0. \end{cases}$$

(a) Compute the distribution of Z.

(b) Compute the correlation between X and Z.

(c) Compute $\mathbb{P}[X + Z = 0]$.

(d) Does (X, Z) follow a multivariate normal distribution (in other words, is (X, Z) a Gaussian vector)?

Exercise 9.5 (Repetition of Exercise 5.3). Let $(S_n)_{n=0,1,\dots,N}$ be a random walk and recall the family $\mathcal{F}_n$ of events observable up to time n. Every $A \in \mathcal{F}_n$ can be written as a union of sets from a partition $\mathcal{G}_n$ of Ω, and we define

$$\mathbb{E}[Z \mid \mathcal{F}_n] := \mathbb{E}[Z \mid \mathcal{G}_n]$$

for any random variable Z. Let $Y_n = \exp(S_n/\sqrt{N})$ for $n = 0, 1, \dots, N$ and define $Z_n := \mathbb{E}[Y_N \mid \mathcal{F}_n]$ for $n = 0, 1, \dots, N$.

(a) Show that $Z_n := Y_n \left(\cosh(1/\sqrt{N})\right)^{N-n}$ for $n = 0, 1, \dots, N$.

(b) Prove that $\mathbb{E}[Z_n \mid \mathcal{F}_m] = Z_m$ for $m \leq n$. (This means that Z is a martingale.)

(c) Prove that $\mathbb{E}\left[S_N^2 | \mathcal{F}_n\right] = S_n^2 + N - n$ for $n = 0, 1, \dots, N$

Hint: Show first that $S_N - S_n$ is independent of $\mathcal{G}_n$. You can use without proof the fact that $X_1, X_2, \dots, X_N$ are i.i.d. for every $N \in \mathbb{N}$, that $\mathcal{F}_n$ is the σ-algebra generated by $X_1, X_2, \dots, X_n$ and hence the independence of $(\mathcal{F}_n, X_{n+1}, X_{n+2}, \dots, X_N)$. Use Exercises 5.1 and 5.2.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.