Probability and Statistics

Exercise sheet 10

Exercise 10.1 Seien $X_2, X_3, \dots$ unabhängige Zufallsvariablen mit $\mathbb{P}[X_n = n] = \mathbb{P}[X_n = -n] = \frac{1}{2n\log n}$ und $\mathbb{P}[X_n = 0] = 1 - \frac{1}{n\log n}$, $n = 2, 3, \dots$. Zeige, dass die Folge $(X_n)_{n\geq 2}$ das schwache, aber nicht das starke Gesetz der grossen Zahlen erfüllt.

Solution 10.1 Wir zeigen zuerst, dass das starke Gesetz der grossen Zahlen nicht gilt. Betrachte die Ereignisse $A_n = \{|X_n| \geq n\}, n \geq 2$. Dann gilt

$$P[A_n] = 1/(n\log n), \text{ also } \sum_{n=2}^{\infty} P[A_n] = \infty.$$

Aus der Divergenz der Reihe und der Unabhängigkeit der X_i folgt mit dem zweiten Teil des Lemmas von Borel-Cantelli, dass die Ereignisse A_n unendlich oft eintreten mit Wahrscheinlichkeit 1. Wenn das starke Gesetz gelten würde, müsste $S_n/n \to 0$ P-f.s. gelten, denn $\mathbb{E}[X_k] = 0$. Dann müsste es zu fast jedem ω und jedem $\epsilon > 0$ ein $n_0(\omega, \epsilon)$ geben, so dass für $n \geq n_0$ gilt $|S_n(\omega)| \leq n\epsilon$. Dann folgt aber für $n \geq n_0$, dass $|X_{n+1}| \leq |S_{n+1}| + |S_n| \leq (n+1)\epsilon + n\epsilon \leq 2(n+1)\epsilon$. Dies ist ein Widerspruch zu dem unendlich oft Eintreten der A_n.

Wir zeigen jetzt, dass das schwache Gesetz der grossen Zahlen gilt. Es gilt $\mathrm{Var}[X_k] = \mathbb{E}\left[X_k^2\right] = k/\log k$, da $\mathbb{E}[X_k] = 0$. Da die Funktion $x/\log x$ auf $[1, \infty)$ ein lokales Minimum bei $x = e$ hat, erhalten wir mit der Chebyshev-Ungleichung für $\epsilon > 0$, dass

$$\begin{aligned} P[|S_n/n| \geq \epsilon] &\leq \frac{1}{\epsilon^2 n^2} \sum_{k=2}^{n} \mathrm{Var}[X_k] \leq \frac{1}{\epsilon^2 n^2}\Big(\frac{2}{\log 2} + \sum_{k=3}^{n} (k/\log k)\Big) \\ &\leq \frac{2}{\epsilon^2 n^2 \log 2} + \frac{(n-3)n}{\epsilon^2 n^2 \log n} \to 0,\ n \to \infty. \end{aligned}$$

Exercise 10.2 We want to estimate the number of a certain type of fish in a lake. For this, we mark 5 fish and we let them mix with the others. When they are well mixed, we fish 11, and we observe that there are 3 marked and 8 non-marked. To estimate the total number of fish N, compute the probabilities of this outcome as a function of N and then take as estimated value N^* for N that N for which this probability becomes maximized. In other words, choose N to give the observed data maximal probability. Which N^* do you find?

Solution 10.2 Define X to be the number of marked fish we fished. If there are N fish in the lake, the probability of $X = 3$ is given by

$$\begin{aligned} \mathbb{P}_N[X = 3] &= \frac{\binom{5}{3}\binom{N-5}{8}}{\binom{N}{11}} I_{\{N\geq 13\}} \\ &= \frac{5!(N-5)!11!(N-11)!}{3!2!8!(N-13)!N!} I_{\{N\geq 13\}} =: g(N). \end{aligned}$$

(The random variable X has a so-called hypergeometric distribution.) We have to find $N_{\max} \in \mathbb{N}$

so that $g(N_{\max}) = \sup_{N\in\mathbb{N}} g(N)$. We have that for $N \geq 13$,

$$\frac{g(N)}{g(N+1)} - 1 = \frac{(N-12)(N+1)}{(N-4)(N-10)} - 1$$
$$= \frac{3(N-17,333\ldots)}{(N-4)(N-10)},$$

thus,

$$\frac{g(N)}{g(N+1)} \begin{cases} \leq 1 & \text{if } N \leq 17, \\ \geq 1 & \text{if } N \geq 18. \end{cases}$$

Then $N^* = 18$.

Exercise 10.3 Let $X_i, \ldots X_n$ be a sequence of i.i.d. Bernoulli-distributed random variables with unknown parameter p. We consider two different estimators. The first estimator is X_1. The second estimator is $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

(a) Compute the expected value and the variance as a function of n for both estimators. Comment on the result.

(b) Are these estimators unbiased?

(c) Are these estimators consistent? (More precisely: Are the sequences of estimators $(X_1)_{n\in\mathbb{N}}$ and $(\overline{X}_n)_{n\in\mathbb{N}}$ consistent?)

Solution 10.3

(a) The expected values are given by $\mathbb{E}[X_1] = p$ and $\mathbb{E}\left[\overline{X}_n\right] = p$. The variances are given by $\mathrm{Var}[X_1] = p(1-p)$ and $\mathrm{Var}\left[\overline{X}_n\right] = \frac{1}{n^2}\sum_{i=1}^n p(1-p) = \frac{p(1-p)}{n}$. Clearly the second estimator is much better as it has much smaller variance and the same expectation.

(b) Both estimators are unbiased estimators for p, because of (a).

(c) The first estimator X_1 is only consistent for $p \in \{0, 1\}$. For $p \in (0, 1)$ this estimator is not consistent, since $\mathbb{P}_p[|X_1 - p| > \varepsilon] = 1$ for $\varepsilon = \min(\frac{p}{2}, \frac{1-p}{2})$ and 1 obviously does not converge to 0. However the second estimator $\overline{X}_n$ is a consistent estimator for p, because (a) showed that $\lim_{n\to\infty} \mathrm{Var}\left[\overline{X}_n\right] = 0$ (This implies convergence to the expected value with the help of the Chebyshev-inequality).

Exercise 10.4 Let $X_i, \ldots X_n$ be a sequence of i.i.d. exponential random variables with unknown parameter α. The so-called moment estimator for α is given by $\hat{\alpha}_n := \frac{1}{\overline{X}_n} = \frac{n}{\sum_{i=1}^n X_i}$.

(a) Is this estimator unbiased?

(b) Is this estimator consistent? (More precisely: Is the sequence of estimators $(\hat{\alpha}_n)_{n\in\mathbb{N}}$ consistent?)

Solution 10.4 First, remember that $\mathbb{E}[X_i] = \frac{1}{\alpha}$ and $\mathrm{Var}[X_i] = \frac{1}{\alpha^2}$. This implies that $\mathbb{E}\left[\overline{X}_n\right] = \frac{1}{\alpha}$ and by independence $\mathrm{Var}\left[\overline{X}_n\right] = \frac{1}{\alpha^2 n}$

(a) We show that $\hat{\alpha}_n$ is biased (not unbiased). When we go through the proof of the Jensen inequality we see quite directly that for a strictly convex function g and a not deterministic random variable X, we get a strict Jensen inequality $\mathbb{E}[g(X)] > g(\mathbb{E}[X])$. The map $x \mapsto \frac{1}{x}$ is strictly convex on $(0,\infty)$ and $\overline{X}_n$ is not deterministic. This leads to the inequality

$$\alpha = \frac{1}{\mathbb{E}\left[\overline{X}_n\right]} < \mathbb{E}\left[\frac{1}{\overline{X}_n}\right] = \mathbb{E}\left[\hat{\alpha}_n\right],$$

which further leads to a bias $b_{\hat{\alpha}_n}(\alpha) = \mathbb{E}[\hat{\alpha}_n] - \alpha > 0$.

(b) We show that $\hat{\alpha}_n$ is a consistent estimator for α. By the strong law of large numbers we obtain that $\overline{X}_n$ converges in almost surely to $\frac{1}{\alpha}$. Since the map $x \mapsto \frac{1}{x}$ is continuous on $(0,\infty)$, we get that $\hat{\alpha}_n := \frac{1}{\overline{X}_n}$ converges almost surely to $\dfrac{1}{\frac{1}{\alpha}} = \alpha$, as $\{\omega \in \Omega : \lim_{n\to\infty} \overline{X}_n = \frac{1}{\alpha}\} \subseteq \{\omega \in \Omega : \lim_{n\to\infty} g(\overline{X}_n) = g(\frac{1}{\alpha})\}$ for any continous function g. Almost sure convergence implies stochastic convergence and hence $\hat{\alpha}$ is consistent.

(A biased sequence of estimators can be consistent, if the bias converges to 0 as n tends to infinity, also called *asymptotically unbiased*.)

Alternative solution: We can directly see with the Chebyshev-inequality that $\overline{X}_n$ converges in probability to $\frac{1}{\alpha}$. Since the map $x \mapsto \frac{1}{x}$ is continuous on $(0,\infty)$, we get that $\hat{\alpha}_n := \frac{1}{\overline{X}_n}$ converges in probability to $\dfrac{1}{\frac{1}{\alpha}} = \alpha$, which makes $\hat{\alpha}$ consistent.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.