Probability and Statistics

Exercise sheet 2

Please ask questions in the exercise classes and/or post your questions (anonymously if you want) in this file: https://docs.google.com/document/d/1xObbQFXf6GsiarNAJ46pkpOhbzPM5PPXDqkZxsoFA10/edit?usp=sharing

Exercise 2.1 Let $X : \Omega \to \mathbb{R}_+$ be a nonnegative discrete random variable taking its values in the set $\{x_1, x_2, \ldots\}$ (possibly countably infinite), where we assume that the values are ordered as $x_1 < x_2 < \cdots$. Show that

$$\mathbb{E}[X] = \sum_{j=0}^{\infty}(x_{j+1} - x_j)\mathbb{P}[X > x_j] \tag{1}$$

with $x_0 := 0$.

How does this connect to equation (1.12) from the lecture notes?

Solution 2.1 We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{\infty}(x_k - x_0)\mathbb{P}[X = x_k] \\
&= \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}(x_{j+1} - x_j)\mathbb{P}[X = x_k] \\
&= \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}(x_{j+1} - x_j)\mathbb{P}[X = x_k]\, I_{\{j \le k-1\}} \\
&= \sum_{j=0}^{\infty}\sum_{k=1}^{\infty}(x_{j+1} - x_j)\mathbb{P}[X = x_k]\, I_{\{k \ge j+1\}} \\
&= \sum_{j=0}^{\infty}(x_{j+1} - x_j)\sum_{k=1}^{\infty}\mathbb{P}[X = x_k]\, I_{\{k \ge j+1\}} \\
&= \sum_{j=0}^{\infty}(x_{j+1} - x_j)\sum_{k=j+1}^{\infty}\mathbb{P}[X = x_k] \\
&= \sum_{j=0}^{\infty}(x_{j+1} - x_j)\mathbb{P}[X \ge x_{j+1}] \\
&= \sum_{j=0}^{\infty}(x_{j+1} - x_j)\mathbb{P}[X > x_j]
\end{aligned}$$

One must be careful to justify exchanging the summations, but this is correct in this case as a classical result from analysis, since all the summands are positive.

Equation (1) matches equation (1.12) from the lecture notes for integer-valued random variables, where $\{x_1, x_2, ...\} = \{0, 1, ...\}$, so that $x_{j+1} - x_j = 1$ for all $j \ge 2$.

Exercise 2.2 In a building with 6 floors (plus the ground floor), an elevator starts with 4 people at the ground floor. What is the probability that these people get off at exactly 2 floors?

Solution 2.2 If P_i $(i = 1, ..., 4)$ are the 4 people, each has 6 possibilities of where to get off. Thus, we take $\Omega = \{1, 2, 3, 4, 5, 6\}^4$ so that $|\Omega| = 6^4$, and we take as $\mathbb{P}$ the uniform distribution (see section 1.2 in the lecture notes). Write

$$A = \{P_1, P_2, P_3, P_4 \text{ get off at exactly 2 floors}\}.$$

To compute $|A|$, there are $\binom{6}{2}$ ways of selecting those two floors. Then, given the 2 chosen floors, there are $\binom{4}{1} + \binom{4}{2} + \binom{4}{3}$ ways of distributing people between those floors, because one floor will have 1 or 2 or 3 people and the other floor the rest. Therefore,

$$\begin{aligned} \mathbb{P}[A] &= \frac{|A|}{|\Omega|} = \frac{\left(\binom{4}{1} + \binom{4}{2} + \binom{4}{3}\right)\binom{6}{2}}{6^4} \\ &= \frac{(4+6+4)\frac{6\times 5}{2}}{1296} \\ &= \frac{14 \times 15}{1296} \\ &= \frac{35}{216} \\ &\approx 0.162. \end{aligned}$$

Exercise 2.3 Consider Beispiel 2.1. Garderobenproblem from the lecture notes, where n coats are distributed randomly to n persons. We assume a Laplace model on the set Ω of all permutations of $\{1, \dots, n\}$ as explained in the lecture notes. What is the expected number of persons who get their own coat back—i.e. $\mathbb{E}[X]$ with $X(\omega) := |\{i \in \{1, \dots, n\} : \omega(i) = i\}|$?
Hint: Use the properties of expectation. Do not try to find the distribution of X.

Solution 2.3 The event $A_i := \{\omega \in \Omega : \omega(i) = i\}$ means that the i-th person getting their own coat back. Since

$$X = \sum_{i=1}^{n} I_{A_i},$$

with the indicator function as in the lecture notes, we can simply use the linearity of the expectation to get

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[I_{A_i}] = \sum_{i=1}^{n} \mathbb{P}[A_i].$$

The probability $\mathbb{P}[A_i]$ on the right-hand side can be computed either by counting $\frac{|\{\omega:\omega(i)=i\}|}{|\Omega|} = \frac{(n-1)!}{n!} = \frac{1}{n}$ or by a simple symmetry argument: It is clear that $\{\omega : \omega(i) = i\} = (n-1)! = \{\omega : \omega(i) = j\}$ $\forall i, j \in \{1, \dots, n\}$ and therefore

$$\mathbb{P}[\{\omega : \omega(i) = i\}]\, n = \mathbb{P}\left[\dot{\bigcup_{j=1}^{n}} \{\omega : \omega(i) = j\}\right] = \mathbb{P}[\{\omega : \omega(i) \in \{1, \dots, n\}\}] = 1.$$

So we can conclude that

$$\mathbb{E}[X] = \sum_{i=1}^{n} \frac{1}{n} = 1,$$

i.e., on average, one person gets their coat back. This does not depend on the number of persons.

Exercise 2.4 One coin is flipped and one die is rolled.

(a) Define a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ space using a Laplace model.

(b) Define random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ on this probability space such that X and Y represent the outcome of flipping the coin and of the roll of the die, respectively.

(c) Show that $\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\,\mathbb{P}[Y = y]$ for all $x, y \in \mathbb{R}$. (This means that the random variables X and Y are independent; see later.)

Solution 2.4

(a) $\Omega := \{0,1\} \times \{1,2,3,4,5,6\}$, $\mathcal{F} := 2^{\Omega}$, $\mathbb{P} : \mathcal{F} \to [0,1], A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{12}$.

(b) $X : \Omega \to \mathbb{R}, \omega \mapsto X(\omega) := \omega_1$ and $Y : \Omega \to \mathbb{R}, \omega \mapsto Y(\omega) := \omega_2$.

(c) First we compute the left-hand side as

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}] = \frac{|\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}|}{12},$$

where the numerator can be simplified as

$$|\{\omega \in \Omega : X(\omega) = x \text{ and } Y(\omega) = y\}| = \begin{cases} 1, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

This results in

$$\mathbb{P}[X = x, Y = y] = \begin{cases} \frac{1}{12}, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

Now, we consider the right-hand side by starting with calculating the probability

$$\mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}] = \frac{|\{\omega \in \Omega : X(\omega) = x\}|}{12},$$

where the set appearing in the numerator can be expressed as

$$\{\omega \in \Omega : X(\omega) = x\} = \begin{cases} \{(x,1),(x,2),(x,3),(x,4),(x,5),(x,6)\}, & x \in \{0,1\}, \\ \emptyset, & x \notin \{0,1\}. \end{cases}$$

So by counting, we obtain

$$\mathbb{P}[X = x] = \begin{cases} \frac{6}{12} = \frac{1}{2}, & x \in \{0,1\}, \\ 0, & x \notin \{0,1\}. \end{cases}$$

Analogously, we get

$$\mathbb{P}[Y = y] = \begin{cases} \frac{2}{12} = \frac{1}{6}, & y \in \{1,2,3,4,5,6\}, \\ 0, & y \notin \{1,2,3,4,5,6\}. \end{cases}$$

Since $(x,y) \in \Omega$ is equivalent to $(x \in \{0,1\} \ \wedge \ y \in \{1,2,3,4,5,6\})$, we can conclude that

$$\mathbb{P}[X = x]\,\mathbb{P}[Y = y] = \begin{cases} \frac{1}{2}\frac{1}{6} = \frac{1}{12}, & (x,y) \in \Omega, \\ 0, & (x,y) \notin \Omega. \end{cases}$$

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.