Probability and Statistics

Exercise sheet 3

Exercise 3.1 We have two dice. One is ordinary with the numbers $1, 2, 3, 4, 5, 6$ and one is special where 6 is replaced by 7 (i.e. $1, 2, 3, 4, 5, 7$). We flip a coin to decide which die is rolled. If flipping the coin results in heads, the ordinary die is rolled, otherwise the special die is rolled.

(a) Define a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ using a Laplace model.

(i) Define random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ such that X and Y represent the outcomes of flipping the coin and of the roll of the die, respectively.

(ii) What is the cardinality $|\mathcal{F}|$? Give examples of events $E_1, E_2, E_3, E_4 \in \mathcal{F}$ such that $\mathbb{P}[E_i] \neq \mathbb{P}[E_j]$, $\forall i \neq j$.

(b) What is the probability that rolling the die results in an even number?

(c) Show that $\mathbb{P}[X = x, Y = y] \neq \mathbb{P}[X = x]\,\mathbb{P}[Y = y]$ for some $x, y \in \mathbb{R}$. (This means that the random variables X and Y are not independent; see later and cf. Exercise 2.4 from Excercise Sheet 2.)

Solution 3.1

(a) $\Omega := \{0,1\} \times \{1,2,3,4,5,6\}$, $\mathcal{F} := 2^\Omega$, $\mathbb{P} : \mathcal{F} \to [0,1]$, $A \mapsto \mathbb{P}[A] = \frac{|A|}{|\Omega|} = \frac{|A|}{12}$. Alternatively one could define $\Omega := \{(0,1),(0,2),(0,3),(0,4),(0,5),(0,6),(1,1),(1,2),(1,3),(1,4),(1,5),(1,7)\}$. Then Y can be defined more simply as $Y(\omega) := \omega_2$.

(i) $X : \Omega \to \mathbb{R}, \omega \mapsto X(\omega) := \omega_1$ and $Y : \Omega \to \mathbb{R}, \omega \mapsto Y(\omega) := \begin{cases} \omega_2, & \omega_2 \leq 5, \\ 6, & \omega_2 = 6 \text{ and } \omega_1 = 0, \\ 7, & \omega_2 = 6 \text{ and } \omega_1 = 1. \end{cases}$

An elegant way to write this is $Y(\omega) := \omega_2 I_{\{\omega_2 \leq 5\}} + (\omega_1 + \omega_2) I_{\{\omega_2 > 5\}}$.

(ii) $|\mathcal{F}| = 2^{12} = 4096$, so there are many different possibilities to choose the examples—e.g. $E_1 = \emptyset, E_2 = \{(0,1)\}, E_3 = \{(0,1),(0,3),(1,2)\}, E_4 = \Omega$ with $\mathbb{P}[E_1] = 0$, $\mathbb{P}[E_2] = \frac{1}{12}$, $\mathbb{P}[E_3] = \frac{3}{12}$, $\mathbb{P}[E_4] = 1$.

(b) We can directly count that $|\{\omega \in \Omega : \exists k \in \mathbb{N} \text{ with } Y = 2k\}| = 3 + 2 = 5$. Hence, the probability is $\mathbb{P}[\{\omega \in \Omega : \exists k \in \mathbb{N} \text{ with } Y = 2k\}] = \frac{5}{12}$.

(c) To show the dependence of the two random variables, it is sufficient to find one counterexample for the equation. A simple choice is $x = 0$ and $y = 7$. First we compute the left-hand side as

$$\mathbb{P}[X = 0, Y = 7] = \frac{|\{\omega \in \Omega : X(\omega) = 0 \text{ and } Y(\omega) = 7\}|}{12} = 0.$$

Now, we consider the right-hand side by starting with calculating the probability

$$\mathbb{P}[X = 0] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = 0\}] = \frac{|\{\omega \in \Omega : X(\omega) = 0\}|}{12} = \frac{6}{12} = \frac{1}{2}.$$

Analogously, we get

$$\mathbb{P}[Y = 7] = \mathbb{P}[\{\omega \in \Omega : Y(\omega) = 7\}] = \frac{|\{\omega \in \Omega : Y(\omega) = 7\}|}{12} = \frac{1}{12}.$$

So we can conclude that

$$\mathbb{P}[X=0,Y=7]=0\neq\frac{1}{24}=\mathbb{P}[X=0]\,\mathbb{P}[Y=7].$$

Exercise 3.2 Let $(S_n)_{n=0,1,\dots,2N}$ be a simple random walk on $\{-1,1\}^{2N}$. Fix $a,b\in\mathbb{Z}$

(a) Prove that $T_a:=\min\{n\in\mathbb{N}_0 : S_n=a\}\wedge 2N$ is a stopping time.

(b) Prove that $\tau_b:=\min\{n>T_a : S_n=b\}\wedge 2N$ is a stopping time.

(c) Show that $L=\max\{0\leq n\leq 2N : S_n=0\}$ is not a stopping time if $N\geq 1$.

Solution 3.2 Stopping times are defined in Definition 3.9 in the lecture notes. Let $n\in\{0,\dots,2N\}$ and $a,b\in\mathbb{Z}$ be arbitrary but fixed.

(a) We need to show that $\{\omega\in\Omega : T_a(\omega)\leq n\}\in\mathcal{F}_n$, where $\mathcal{F}_n$ is defined in Definition 3.8 in the lecture notes. Recall that $\mathcal{F}_0=\{\emptyset,\Omega\}$. Recall also that for any stopping time T, $\{T\leq 2N\}=\Omega$ is in $\mathcal{F}_{2N}$ (even in $\mathcal{F}_0$).

case 1) $n=2N$: In this case, $\{\omega\in\Omega : T_a(\omega)\leq 2N\}=\Omega\in\mathcal{F}_{2N}$.

case 2) $n\in\{1,\dots,2N-1\}$: According to Definition 3.8, we define a set

$$C:=\left\{x\in\{-1,1\}^n : \exists k\in\{0,\dots,n\}\ \text{ with }\sum_{i=1}^{k}x_i=a\right\}\subseteq\{-1,1\}^n.$$

With this definition, we get

$$\{\omega\in\Omega : T_a(\omega)\leq n\}=\{\omega\in\{-1,1\}^{2N} : (X_1(\omega),\dots,X_n(\omega))\in C\}\in\mathcal{F}_n.$$

case 3) $n=0$: If $a=0$, then $T_a\equiv 0$ and $\{\omega\in\Omega : T_a(\omega)\leq 0\}=\Omega\in\mathcal{F}_0$. If $a\neq 0$, then $T_a(\omega)\neq 0$ for all ω and $\{\omega\in\Omega : T_a\leq 0\}=\emptyset\in\mathcal{F}_0$.

(b) We need to show that $\{\omega\in\Omega : \tau_b(\omega)\leq n\}\in\mathcal{F}_n$, where $\mathcal{F}_n$ is defined in Definition 3.8 in the lecture notes.

case 1) $n=2N$: In this case, $\{\omega\in\Omega : \tau_b(\omega)\leq 2N\}=\Omega\in\mathcal{F}_{2N}$.

case 2) $n\in\{1,\dots,2N-1\}$: According to Definition 3.8, we define a set

$$C:=\left\{x\in\{-1,1\}^n : \exists k<\ell\in\{0,\dots,n\}\ \text{ with }\sum_{i=1}^{k}x_i=a\text{ and }\sum_{i=1}^{\ell}x_i=b\right\}\subseteq\{-1,1\}^n.$$

With this definition, we get

$$\{\omega\in\Omega : \tau_b(\omega)\leq n\}=\{\omega\in\{-1,1\}^{2N} : (X_1(\omega),\dots,X_n(\omega))\in C\}\in\mathcal{F}_n.$$

case 3) $n=0$: Since $T_a\geq 0$ we directly get that $\tau_b>0$, which implies $\{\omega\in\Omega : \tau_b\leq 0\}=\emptyset\in\mathcal{F}_0$.

(c) We prove the result by contradiction. If L were a stopping time, then (3.12) would imply for example that $\{\omega : L(\omega)\leq 0\}\in\mathcal{F}_0=\{\emptyset,\Omega\}$. But we can show that $\{\omega : L(\omega)\leq 0\}$ is neither the empty set $\emptyset$ nor the whole sample space Ω.

We start by showing $\{\omega : L(\omega)\leq 0\}\neq\emptyset$ by writing down explicitly an $\omega^*\in\{\omega : L(\omega)\leq 0\}$, namely $\omega^*=(1,\dots,1)$, since $L(\omega^*)=0$, because $S_n(\omega^*)=n\neq 0,\ \forall n>0$.

We finish the proof by showing $\{\omega : L(\omega)\leq 0\}\neq\Omega$ by writing down explicitly an $\omega^\circ\notin\{\omega : L(\omega)\leq 0\}$, namely $\omega^\circ=(1,-1,1,1,1,\dots,1)$. Then $L(\omega^\circ)=2$ because $S_2(\omega^*)=0$ and $S_n(\omega^\circ)>0,\ \forall n>2$.

Exercise 3.3 Let $(S_n)_{n=0,1,\dots,N}$ be a simple random walk on $\{-1,1\}^N$. Consider the strategy of first betting 1 and then successively tripling your bet until you win for the first time.

(a) Describe this strategy by a gambling strategy V.

(b) Calculate the resulting total income $(V \bullet S)_N$ and its expected value $\mathbb{E}\left[(V \bullet S)_N\right]$.

(c) Calculate the distribution of $(V \bullet S)_N$ and use this to compute $\mathbb{E}\left[(V \bullet S)_N\right]$ again.

(d) Can you find a gambling system V with $(V \bullet S)_N = {S_N}^4$?

Solution 3.3

(a) Set $V_1 := 1$ and recursively $V_{k+1} := 3V_k I_{\{X_k=-1\}}$ so that

$$V_{k+1} = 3^k I_{\{X_1=\cdots=X_k=-1\}} = 3^k I_{\{S_k=-k\}}.$$

This is indeed a gambling system as V_{k+1} only depends on $X_1,\dots,X_k$. With $\tau := \min\{k \in \{1,\dots,N\} : X_k = +1\}$ and $\min\emptyset := N+1$, we can also write $V_k = 3^{k-1} I_{\{k\le\tau\}}$ for $k \in \{1,\dots,N\}$.

(b) By definition,

$$\begin{aligned}(V \bullet S)_N &= \sum_{k=1}^{N} V_k(S_k - S_{k-1}) = \sum_{k=1}^{N} V_k X_k \\ &= \sum_{k=1}^{N} 3^{k-1} X_k I_{\{k\le\tau\}}.\end{aligned}$$

On the set $\{\tau = N+1\} = \{-1,\dots,-1\}$, this gives

$$(V \bullet S)_N = \sum_{k=1}^{N} 3^{k-1}(-1) = -\sum_{i=0}^{N-1} 3^i = -\frac{1}{2}(3^N - 1).$$

On $\{\tau = K \le N\}$, we have analogously

$$\begin{aligned}(V \bullet S)_N &= \sum_{k=1}^{K} 3^{k-1} X_k = \sum_{k=1}^{K-1} 3^{k-1}(-1) + 3^{K-1}(+1) \\ &= -\frac{1}{2}(3^{K-1} - 1) + 3^{K-1} = \frac{1}{2}(3^{K-1} + 1).\end{aligned}$$

We can write this as

$$(V \bullet S)_N = -\frac{1}{2}(3^N - 1) I_{\{\tau>N\}} + \frac{1}{2}(3^{\tau-1} + 1) I_{\{\tau\le N\}}.$$

Because V is a gambling system, we have $\mathbb{E}\left[(V \bullet S)_N\right] = 0$ from Satz 3.17 from the lecture notes.

(c) Because $\mathbb{P}\left[\tau = N+1\right] = 2^{-N}$ and $\mathbb{P}\left[\tau = K \le N\right] = 2^{-K}$, we get

$$\begin{aligned}\mathbb{P}\left[(V \bullet S)_N = -\frac{1}{2}(3^N - 1)\right] &= 2^{-N} \text{ and} \\ \mathbb{P}\left[(V \bullet S)_N = \frac{1}{2}(3^{K-1} + 1)\right] &= 2^{-K} \text{ for } K \in \{1,\dots,N\}.\end{aligned}$$

So we compute

$$
\begin{aligned}
\mathbb{E}\left[(V \bullet S)_N\right] &= -2^{-(N+1)}(3^N - 1) + \sum_{K=1}^{N} 2^{-(K+1)}(3^{K-1} + 1) = \\
&= -\frac{1}{2}\left(\frac{3}{2}\right)^N + \frac{1}{2}2^{-N} + \frac{1}{4}\sum_{K=1}^{N}\left(\frac{3}{2}\right)^{K-1} + \frac{1}{4}\sum_{K=1}^{N} 2^{-(K-1)} \\
&= -\frac{1}{2}\left(\frac{3}{2}\right)^N + \frac{1}{2}2^{-N} + \frac{1}{2}\left(\frac{3}{2}\right)^N - \frac{1}{2} - \frac{1}{2}2^{-N} + \frac{1}{2} = 0.
\end{aligned}
$$

Of course, the result is the same as in (b), but the computations are much more tedious.

(d) No. By Satz 3.17, we have $\mathbb{E}\left[(V \bullet S)_N\right] = 0$ for any gambling system. But ${S_N}^4$ is nonnegative and not identically 0, i.e., $\mathbb{P}\left[{S_N}^4 > 0\right] > 0$. So we must have $\mathbb{E}\left[{S_N}^4\right] > 0$, and no V as desired can exist.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.