Probability and Statistics

Exercise sheet 5

Exercise 5.1 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a discrete probability space and $\mathcal{G} = (B_i)_{i\in I}$ a countable partition of Ω. A discrete random variable X is called *$\mathcal{G}$-measurable* if it can be written as $X = \sum_{i\in I} c_i I_{B_i}$ with (not necessarily distinct) $c_i \in \mathbb{R}$.

Two set systems $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ are called *independent* if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. A discrete random variable Y is called *independent* of $\mathcal{A}$ if the systems $\mathcal{B} := \{\{Y = c\} : c \in \mathbb{R}\}$ and $\mathcal{A}$ are independent.

(a) Suppose that X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$. Show that for any $F : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\mathbb{E}[F(X,Y) \mid \mathcal{G}](\omega) = \mathbb{E}[F(x,Y)]\big|_{x=X(\omega)}, \tag{1}$$

provided that all expectations are well defined.

(b) Deduce that $\mathbb{E}[X \mid \mathcal{G}] = X$ and $\mathbb{E}[Y \mid \mathcal{G}] = \mathbb{E}[Y]$

Solution 5.1

(a) We start by considering the right-hand side of eq. (1), by plugging in the definition of the expectation $\mathbb{E}[F(x,Y)] = \sum_{y_k\in Y(\Omega)} F(x, y_k)\mathbb{P}[Y = y_k]$ into the right-hand side to get

$$\text{RHS} = \sum_{i\in I} \sum_{y_k\in Y(\Omega)} F(c_i, y_k)\mathbb{P}[Y = y_k]\, I_{B_i}(\omega).$$

Then we compute the left-hand side by using first the basic definitions and then the independence in the second to last step, to obtain

$$\begin{aligned}
\text{LHS} &= \sum_{i\in I} \mathbb{E}[F(X,Y) \mid B_i]\, I_{B_i}(\omega) \\
&= \sum_{i\in I} \frac{1}{\mathbb{P}[B_i]} \mathbb{E}[F(X,Y)I_{B_i}]\, I_{B_i}(\omega) \\
&= \sum_{i\in I} \frac{1}{\mathbb{P}[B_i]} \mathbb{E}[F(c_i,Y)I_{B_i}]\, I_{B_i}(\omega) \\
&= \sum_{i\in I} \frac{1}{\mathbb{P}[B_i]} \sum_{y_k\in Y(\Omega)} F(c_i, y_k)\mathbb{P}[\{Y = y_k\} \cap B_i]\, I_{B_i}(\omega) \\
&= \sum_{i\in I} \frac{1}{\mathbb{P}[B_i]} \sum_{y_k\in Y(\Omega)} F(c_i, y_k)\mathbb{P}[\{Y = y_k\}]\,\mathbb{P}[B_i]\, I_{B_i}(\omega) \\
&= \sum_{i\in I} \sum_{y_k\in Y(\Omega)} F(c_i, y_k)\mathbb{P}[\{Y = y_k\}]\, I_{B_i}(\omega).
\end{aligned}$$

We see that both sides of eq. (1) are equal.

(b) We use eq. (1) with $F_1(X,Y) = X$ to obtain that

$$\mathbb{E}[X \mid \mathcal{G}](\omega) = \mathbb{E}[F_1(x,Y)]\big|_{x=X(\omega)} = \mathbb{E}[x]\big|_{x=X(\omega)} = x\big|_{x=X(\omega)} = X(\omega).$$

To prove the other identity, we use eq. (1) with $F_2(X,Y) = Y$ to obtain that

$$\mathbb{E}\left[Y \mid \mathcal{G}\right](\omega) = \mathbb{E}\left[F_2(x,Y)\right]\Big|_{x=X(\omega)} = \mathbb{E}\left[Y\right]\Big|_{x=X(\omega)} = \mathbb{E}\left[Y\right].$$

Exercise 5.2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a discrete probability space, $\mathcal{G} = (B_i)_{i\in I}$ a countable partition of Ω and $(X_k)_{k\in\{1,\ldots,n\}}$ a family of discrete random variables. Let the collection of $(\mathcal{G}, X_1, X_2, \ldots, X_n)$ be independent, i.e. the events $(B_i, \{X_1 = x_1\}, \{X_2 = x_2\}, \ldots, \{X_n = x_n\})$ are independent for all $B_i \in \mathcal{G}$ and $x_k \in X_k(\Omega)$. Show that $Y : \Omega \to \mathbb{R}, \omega \mapsto f(X_1(\omega), \ldots, X_n(\omega))$ is independent of $\mathcal{G}$ for every $f : \mathbb{R}^n \to \mathbb{R}$.

Solution 5.2 Note that any $x \in f^{-1}(\{c\})$ is an element of $\mathbb{R}^n$. For any $c \in \mathbb{R}$ and $A \in \mathcal{G}$, we have

$$\begin{aligned}
\mathbb{P}\left[\{Y = c\} \cap A\right] &= \mathbb{P}\left[\left(\dot{\bigcup_{x\in f^{-1}(\{c\})}} \bigcap_{k=1}^{n} \{X_k = x_k\}\right) \cap A\right] \\
&= \mathbb{P}\left[\dot{\bigcup_{x\in f^{-1}(\{c\})}} \left(\bigcap_{k=1}^{n} \{X_k = x_k\} \cap A\right)\right] \\
&= \sum_{x\in f^{-1}(\{c\})} \mathbb{P}\left[\left(\bigcap_{k=1}^{n} \{X_k = x_k\} \cap A\right)\right] \\
&= \sum_{x\in f^{-1}(\{c\})} \mathbb{P}\left[\bigcap_{k=1}^{n} \{X_k = x_k\}\right] \mathbb{P}\left[A\right] \\
&= \mathbb{P}\left[Y = c\right] \mathbb{P}\left[A\right].
\end{aligned}$$

Exercise 5.3 Let $(S_n)_{n=0,1,\ldots,N}$ be a random walk and recall the family $\mathcal{F}_n$ of events observable up to time n. Every $A \in \mathcal{F}_n$ can be written as a union of sets from a partition $\mathcal{G}_n$ of Ω, and we define

$$\mathbb{E}\left[Z \mid \mathcal{F}_n\right] := \mathbb{E}\left[Z \mid \mathcal{G}_n\right]$$

for any random variable Z. Let $Y_n = \exp\left(S_n/\sqrt{N}\right)$ for $n = 0, 1, \ldots, N$ and define $Z_n := \mathbb{E}\left[Y_N \mid \mathcal{F}_n\right]$ for $n = 0, 1, \ldots, N$.

(a) Show that $Z_n := Y_n \left(\cosh\left(1/\sqrt{N}\right)\right)^{N-n}$ for $n = 0, 1, \ldots, N$.

(b) Prove that $\mathbb{E}\left[Z_n \mid \mathcal{F}_m\right] = Z_m$ for $m \le n$. (This means that Z is a martingale.)

Hint: Show first that $S_N - S_n$ is independent of $\mathcal{G}_n$.

Solution 5.3

(a) First we show that the collection of $(\mathcal{F}_n, X_{n+1}, X_{n+2}, \ldots, X_N)$ is independent. Indeed for all

$A \in \mathcal{F}_n$ and $x_k \in X_k(\Omega)$, we get

$$\begin{aligned}
\mathbb{P}\left[A \cap \bigcap_{k=n+1}^{N} \{X_k = x_k\}\right] &= \frac{|A|}{2^N} \\
&= \frac{|A|}{2^n} \prod_{k=n+1}^{N} \frac{1}{2} \\
&= \frac{|A| 2^{N-n}}{2^N} \prod_{k=n+1}^{N} \frac{2^{N-1}}{2^N} \\
&= \mathbb{P}[A] \prod_{k=n+1}^{N} \mathbb{P}[\{X_k = x_k\}].
\end{aligned}$$

As $\mathcal{G}_n \subseteq \mathcal{F}_n$, $(\mathcal{G}_n, X_{n+1}, X_{n+2}, \ldots, X_N)$ is independent as well. This can be used together with Exercise 5.2 to deduce the hint and show that $S_N - S_n$ is independent of $\mathcal{G}_n$. Now, we can proceed with the help of eq. (1) to get

$$\begin{aligned}
Z_n(\omega) &= \mathbb{E}\left[\exp\left(\frac{S_N}{\sqrt{N}}\right) \middle| \mathcal{G}_n\right](\omega) \\
&= \mathbb{E}\left[\exp\left(\frac{S_N - S_n + S_n}{\sqrt{N}}\right) \middle| \mathcal{G}_n\right](\omega) \\
&= \mathbb{E}\left[\exp\left(\frac{S_N - S_n + s}{\sqrt{N}}\right)\right]\Bigg|_{s=S_n(\omega)} \\
&= \exp\left(\frac{s}{\sqrt{N}}\right)\Bigg|_{s=S_n(\omega)} \mathbb{E}\left[\exp\left(\frac{S_N - S_n}{\sqrt{N}}\right)\right] \\
&= \exp\left(\frac{S_n(\omega)}{\sqrt{N}}\right) \mathbb{E}\left[\prod_{k=n+1}^{N} \exp\left(\frac{X_k}{\sqrt{N}}\right)\right] \\
&= Y_n(\omega) \prod_{k=n+1}^{N} \mathbb{E}\left[\exp\left(\frac{X_k}{\sqrt{N}}\right)\right] \\
&= Y_n(\omega) \prod_{k=n+1}^{N} \frac{1}{2}\left(\exp\left(\frac{-1}{\sqrt{N}}\right) + \exp\left(\frac{+1}{\sqrt{N}}\right)\right) \\
&= Y_n(\omega) \prod_{k=n+1}^{N} \cosh\left(1/\sqrt{N}\right) \\
&= Y_n(\omega) \left(\cosh\left(1/\sqrt{N}\right)\right)^{N-n}.
\end{aligned}$$

(b) First note that vector space of all $\mathcal{G}_m$-measurable random variables is a subspace of the space of all $\mathcal{G}_n$-measurable functions since $\mathcal{G}_m \subseteq \mathcal{G}_n$. Since the concatenation of orthogonal projections on sub spaces $\mathcal{G}_m, \mathcal{G}_n$ with $\mathcal{G}_m \subseteq \mathcal{G}_n$ is equal to the orthogonal projection on the smaller sub space $\mathcal{G}_m$ we obtain

$$\mathbb{E}[Z_n \mid \mathcal{F}_m] = \mathbb{E}\left[\mathbb{E}[Y_N \mid \mathcal{G}_n] \,\middle|\, \mathcal{G}_m\right] = \mathbb{E}[Y_N \mid \mathcal{F}_m] = Z_m.$$

Alternatively, the same argument as in (a) gives for $m < n$ that

$$\mathbb{E}[Y_n \mid \mathcal{F}_m] = \mathbb{E}[Y_n \mid \mathcal{G}_m] = Y_m \left(\cosh\left(1/\sqrt{N}\right)\right)^{n-m} = Z_m \left(\cosh\left(1/\sqrt{N}\right)\right)^{-(N-n)}$$

and so

$$\begin{aligned}\mathbb{E}\left[Z_n \mid \mathcal{F}_m\right] &= \mathbb{E}\left[Y_n \left(\cosh\left(1/\sqrt{N}\right)\right)^{(N-n)} \mid \mathcal{F}_m\right] \\ &= \left(\cosh\left(1/\sqrt{N}\right)\right)^{(N-n)} \mathbb{E}\left[Y_n \mid \mathcal{F}_m\right] = Z_m.\end{aligned}$$

Exercise 5.4 In a clinical trial with two treatment groups, the probability of success in one treatment group is 0.5, and the probability of success in the other is 0.6. Suppose that there are five patients in each group. Assume that the outcomes of all patients are independent.

(a) Calculate the probability that the first group will have at least as many successes as the second group.

(b) Write the solution in a general formula, when both groups have size n and success probabilities p_1 respectively p_2.

Solution 5.4

(a) The number of successes in the first group follows the binomial law $\mathrm{Bin}(5, 0.5)$, and in the second group follows $\mathrm{Bin}(5, 0.6)$. The probability to have $k \in \{0, 1, 2, 3, 4, 5\}$ successes in the two cases are respectively

$$\begin{aligned}\mathbb{P}\left[X_1 = k\right] &= \binom{5}{k} 2^{-5} \quad \text{for the first group} \\ \mathbb{P}\left[X_2 = k\right] &= \binom{5}{k} 3^k 2^{5-k} 5^{-5} \quad \text{for the second group}\end{aligned}$$

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$p=0.5$	0.0312	0.1562	0.3125	0.3125	0.1562	0.0312
$p=0.6$	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

The probability that the first group has at least as many successes as the second group is given by

$$0.03120 * 0.0102 + 0.1562 * (0.0102 + 0.0768) + ... + 0.0312 * 1 = 0.49553028.$$

(b) In general, we have $X_i \sim \mathrm{Bin}(n, p_i)$ wich are independent and want to compute

$$\begin{aligned}\mathbb{P}\left[X_1 \geq X_2\right] &= \sum_{k=0}^{n} \mathbb{P}\left[X_1 = k, X_2 \leq k\right] \\ &= \sum_{k=0}^{n} \mathbb{P}\left[X_1 = k\right] \mathbb{P}\left[X_2 \leq k\right] \\ &= \sum_{k=0}^{n} \binom{n}{k} p_1^k (1-p_1)^{n-k} \sum_{j=0}^{k} \binom{n}{j} p_2^j (1-p_2)^{n-j}.\end{aligned}$$

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.