Probability and Statistics

Exercise sheet 7

Exercise 7.1 Let $X_1, \ldots, X_n$ be i.i.d. with distribution function (cdf) F.

(a) Let $S_n := \max_{1\leq i\leq n} X_i$. Find the cdf of S_n as a function of F.

(b) Do the same for $I_n := \min_{1\leq i\leq n} X_i$.

(c) Fix $x \in \mathbb{R}$ such that $F(x) \in (0,1)$. What is the limit of the cdf of S_n at x as $n \to \infty$? What about the cdf of I_n? How would you interpret these results? What does this mean if $X_1, \ldots, X_n$ take values in a finite set $\{\xi_1, \ldots, \xi_k\}$? To analyse the last question, compute $\mathbb{P}[|S_n - \xi_k| > \delta]$ and $\mathbb{P}[|I_n - \xi_1| > \delta]$ for $\delta > 0$.

Solution 7.1

(a) For $x \in \mathbb{R}$, by using independence and then the identical distribution,

$$\begin{aligned}
\mathbb{P}[S_n \leq x] &= \mathbb{P}\left[\max_{1\leq i\leq n} X_i \leq x\right] \\
&= \mathbb{P}[X_1 \leq x, \ldots, X_n \leq x] \\
&= \prod_{i=1}^{n} \mathbb{P}[X_i \leq x] \\
&= \big(F(x)\big)^n.
\end{aligned}$$

(b) For $x \in \mathbb{R}$, in the same way,

$$\begin{aligned}
\mathbb{P}[I_n \leq x] &= 1 - \mathbb{P}[I_n > x] \\
&= 1 - \mathbb{P}[X_1 > x, \ldots, X_n > x] \\
&= 1 - \prod_{i=1}^{n} \mathbb{P}[X_i > x] \\
&= 1 - \big(1 - F(x)\big)^n.
\end{aligned}$$

(c) Let $x \in \mathbb{R}$ be such that $F(x) \in (0,1)$. Then

$$\lim_{n\to\infty} \mathbb{P}[S_n \leq x] = \lim_{n\to\infty} \big(F(x)\big)^n = 0$$

and

$$\lim_{n\to\infty} \mathbb{P}[I_n \leq x] = \lim_{n\to\infty} \Big(1 - \big(1 - F(x)\big)^n\Big) = 1 - 0 = 1.$$

This can be interpreted as saying that as n grows, the maximum and minimum are dragged to an extreme value (if they stay somewhere inside the support of $X_1, \ldots, X_n$, we obtain values of the cdf of S_n or I_n away from 0 and 1, respectively).

In the example where $X_i \in \{\xi_1, \dots, \xi_k\}$ with $\xi_1 < \cdots < \xi_k$, S_n and I_n converge to ξ_k and ξ_1 respectively, in probability (a concept to be defined later). Indeed,

$$\mathbb{P}\left[|S_n - \xi_k| > \delta\right] = \mathbb{P}\left[S_n < \xi_k - \delta\right] = \big(F(\xi_k - \delta)\big)^n \to 0$$

by (a), and analogously, $\mathbb{P}\left[|I_n - \xi_1| > \delta\right] \to 0$ as $n \to \infty$ by (b), for any $\delta > 0$.

Exercise 7.2

(a) Construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ with

$$\sum_{n\in\mathbb{N}} \mathbb{P}\left[A_n\right] = \infty$$

and $\mathbb{P}\left[\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right] = 0$.

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Take a sequence $(U_n)_{n\in\mathbb{N}}$ of i.i.d. random variables with uniform distribution $\mathcal{U}(0,1)$.

(i) Show that

$$\mathbb{P}\left[\exists \alpha > 1 : \liminf_{n\to\infty} n^\alpha U_n \in \mathbb{R}\right] = 0.$$

Hint: It may be useful to define $A_n^\alpha := \{U_n < n^{-\alpha}\}$ for $\alpha > 1$. Remember that the countable union of sets of probability 0 has probability 0.

(ii) Prove that

$$\mathbb{P}\left[\liminf_{n\to\infty} nU_n \in \mathbb{R}\right] > 0.$$

Solution 7.2

(a) Take $([0,1], \mathcal{B}([0,1]), \lambda)$ as a probability space, where $\mathcal{B}([0,1])$ is the Borel σ-algebra on $[0,1]$ and $\mathbb{P} = \lambda$ the restriction of the Lebesgue measure to $[0,1]$. Let U be the identity function; then U is distributed as a uniform random variable on $[0,1]$ under λ. Define

$$A_n := \left\{x \in [0,1] : U(x) \in \left[0, \frac{1}{n}\right]\right\} = \left[0, \frac{1}{n}\right].$$

Then we have that $\mathbb{P}\left[A_n\right] = \frac{1}{n}$, so $\sum_{n\in\mathbb{N}} \mathbb{P}\left[A_n\right] = \infty$. But we also have $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$ if and only if $x = 0$. Therefore $\mathbb{P}\left[\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right] = 0$.

(b) (i) We use the Borel–Cantelli lemma. Define $A_n^\alpha := \{U_n < n^{-\alpha}\}$; then

$$\sum_{n=1}^{\infty} \mathbb{P}\left[A_n^\alpha\right] = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

and by the Borel–Cantelli lemma, $\mathbb{P}\left[\bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha\right] = 0$. Thus

$$\mathbb{P}\left[\bigcup_{\substack{\alpha>1 \\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha\right] = 0.$$

Let $\omega \in \Omega$ be such that there exists $\alpha(\omega) > 1$ for which $\liminf_{n\to\infty} n^{\alpha(\omega)} U_n(\omega) < \infty$. Then take $1 < \tilde{\alpha}(\omega) < \alpha(\omega)$ with $\tilde{\alpha}(\omega) \in \mathbb{Q}$. We have that $\liminf_{n\to\infty} n^{\tilde{\alpha}(\omega)} U_n(\omega) = 0$. Thus for all $n \in \mathbb{N}$, there exists $m(\omega) > n$ such that $m^{\tilde{\alpha}(\omega)} U_m(\omega) < 1$. Thus, $\omega \in \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha$. Finally this gives

$$\{\exists \alpha > 1 : \liminf_{n\to\infty} n^\alpha U_n \in \mathbb{R}\} \subseteq \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha,$$

and this implies

$$\mathbb{P}\left[\exists \alpha > 1 : \liminf_{n\to\infty} n^\alpha U_n \in \mathbb{R}\right] = 0.$$

(To be precise, we did not show that $B := \{\exists \alpha > 1 : \liminf_{n\to\infty} n^\alpha U_n \in \mathbb{R}\}$ is in $\mathcal{F}$; so we have only argued that B has outer $\mathbb{P}$-measure 0. The conclusion becomes rigorous if we complete $\mathcal{F}$ with respect to $\mathbb{P}$, which means here that we take the Lebesgue σ-algebra instead of the Borel σ-algebra.)

(ii) We use again the Borel–Cantelli lemma. Define $A_n = \{U_n \leq n^{-1}\}$ so that $(A_n)_{n\in\mathbb{N}}$ are independent because the U_n are. Moreover, $\mathbb{P}[A_n] = \frac{1}{n}$ and hence $\sum_{n\in\mathbb{N}} \mathbb{P}[A_n] = \infty$. By Borel–Cantelli,

$$\mathbb{P}\left[\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right] = 1 > 0.$$

In addition, if $\omega \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$, then for all $n \in \mathbb{N}$ there exists $k_n(\omega) \geq n$ such that $k_n(\omega) U_{k_n(\omega)}(\omega) \leq 1$. Thus, $0 \leq \liminf_{n\to\infty} nU_n(\omega) \leq 1$. To conclude,

$$\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \subseteq \left\{\liminf_{n\to\infty} nU_n \in \mathbb{R}\right\} \quad \text{and} \quad \mathbb{P}\left[\liminf_{n\to\infty} nU_n \in \mathbb{R}\right] = 1 > 0.$$

Exercise 7.3

(a) Let $X \sim \mathcal{U}(0,1)$. Compute $\mathbb{E}[X^n]$, $\mathbb{E}\left[X^{\frac{1}{n}}\right]$ for $n \in \mathbb{N}$, and $\Psi_X(t) := \mathbb{E}\left[e^{tX}\right]$ whenever these are defined. For which t is this the case?

(b) Let $X \sim \mathrm{Exp}(\alpha)$ for $\alpha > 0$. Derive the cdf of X and $\mathbb{E}[X^n]$ for $n \geq 1$ from the density function.

Remark: Watch out for the parametrisation—different sources use different parametrisations. In the lecture we consider the density function of an $\mathrm{Exp}(\alpha)$-distributed random variable to be $f(x) = \alpha e^{-\alpha x}$ for $x \geq 0$.

Solution 7.3

(a) For $\alpha > -1$,

$$\mathbb{E}[X^\alpha] = \int_0^1 x^\alpha dx = \frac{1}{\alpha+1}.$$

In particular,

$$\mathbb{E}[X^n] = \frac{1}{n+1}$$

and

$$\mathbb{E}\left[X^{\frac{1}{n}}\right] = \frac{n}{n+1}.$$

Moreover,

$$\Psi_X(t) = \mathbb{E}\left[e^{tX}\right] = \int_0^1 e^{tx}dx = \frac{e^t - 1}{t}$$

for any $t \neq 0$, while $\Psi_X(0) = 1$. In particular, $\Psi_X(t)$ is well defined for any $t \in \mathbb{R}$. (This can also be seen directly because e^{tX} is bounded by $1 + e^t$ for any fixed $t \in \mathbb{R}$.)

(b) The density of X is

$$f(x) = \alpha e^{-\alpha x} I_{\{x \geq 0\}}.$$

For $t > 0$, we calculate

$$\int_0^t \alpha e^{-\alpha x}dx = 1 - e^{-\alpha t}$$

and therefore

$$F(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\alpha t}, & t \geq 0. \end{cases}$$

We also compute, with the substitution $y = \alpha x$

$$\begin{aligned} \mathbb{E}\left[X^n\right] &= \int_0^\infty x^n \alpha e^{-\alpha x} dx \\ &= \int_0^\infty \frac{y^n}{\alpha^n} \alpha e^{-y} \frac{dy}{\alpha} \\ &= \frac{1}{\alpha^n} \int_0^\infty y^n e^{-y} dy \\ &= \frac{\Gamma(n+1)}{\alpha^n} \\ &= \frac{n!}{\alpha^n} \end{aligned}$$

(using properties of the gamma function; alternatively one could integrate by parts n times—this would directly lead to $n!$ instead of $\Gamma(n+1)$).

Exercise 7.4 An auto towing company services a 50 mile stretch of a highway. The company is located 20 miles from one end of the stretch, but inside the stretch. Breakdowns occur uniformly along the highway, and the towing trucks travel at a constant speed of 50mph. Find the mean and variance of the time elapsed between the instant the company is called and the instant a towing truck arrives at the breakdown.

Where is the optimal location for the company if they want to minimize the expected waiting time?

Solution 7.4 Call the left endpoint of the 50 mile stretch zero, and let X be the number of miles from the left endpoint that a breakdown occurs. Then $X \sim \mathcal{U}(0, 50)$. Assume that the towing company is located 20 miles from the left endpoint, so that the distance Y of the breakdown from the location of the towing company is $Y = |X - 20|$. It will take the truck $Z = \frac{Y}{50} = \frac{|X-20|}{50}$ hours to reach the location of the breakdown. We want the mean and variance of Z. First,

$$\begin{aligned}\mathbb{E}[Z] &= \mathbb{E}\left[\frac{|X-20|}{50}\right] = \frac{1}{50}\int_0^{50}|x-20|f(x)dx = \frac{1}{50^2}\int_0^{50}|x-20|dx \\ &= \frac{1}{2500}\left(\int_0^{20}(20-x)dx + \int_{20}^{50}(x-20)dx\right) = \frac{1}{2500}(200+450) = 0.26 \text{ (hours)}.\end{aligned}$$

Next,

$$\begin{aligned}\mathbb{E}\left[Z^2\right] &= \frac{1}{2500}\mathbb{E}\left[(X-20)^2\right] = \frac{1}{2500}\mathbb{E}\left[X^2-40X+400\right] \\ &= \frac{1}{2500}\frac{1}{50}\int_0^{50}(x^2-40x+400)dx \approx 0.0933 \text{ (hours}^2\text{)},\end{aligned}$$

and therefore

$$\operatorname{Var}[Z] = \mathbb{E}\left[Z^2\right] - \mathbb{E}[Z]^2 = 0.0933 - 0.26^2 \approx 0.0257 \text{ (hours}^2\text{)}.$$

(If one wanted the units, one could write $\operatorname{Var}[Z] \approx 0.0257h^2$). This gives the standard deviation $\sigma(Z) \approx 0.16$ (hours)

The optimal location is the median of X, because of Exercise 6.4(d). Since we assumed that $X \sim \mathcal{U}(0,50)$, the median of X is 25. So the optimal location is in the middle of the stretch, as one expects from intuition or symmetry.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.