Probability and Statistics

Exercise sheet 9

Exercise 9.1 Let $X_1, X_2, \ldots, X_n$ be i.i.d. Cauchy-distributed, i.e. with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. Show that $\bar{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$ is again Cauchy-distributed using two different approaches:

(a) Use the convolution formula (2.2.4) to prove the claim.

Hint: Calculating the integral that appears in this proof would require a tedious partial fraction decomposition or the use of the residue theorem. You can skip these calculations by using

$$\int_{-\infty}^{+\infty} \frac{1}{(a^2+y^2)(b^2+(x-y)^2)}dy = \frac{\pi(a+b)}{ab\left((a+b)^2+x^2\right)}.$$

(b) Use characteristic functions to prove the claim.

(c) How does this fit together with the weak law of large numbers, the strong law of large numbers and the central limit theorem?

Solution 9.1

(a) We prove the statement by induction. For $n = 1$, the statement is trivial.

Before we continue, we note in general that if Y has density f_Y and $a > 0$, then aY has density $f_{aY}(y) = \frac{1}{a}f_Y(y/a)$ because the distribution functions are related via $F_{aY}(y) = \mathbb{P}[aY \le y] = F_Y(y/a)$. So if $X \sim$ Cauchy, then aX has the density $f_{aX}(x) = \frac{1}{\pi}\frac{a}{a^2+x^2}$. For the induction step, we need to show that $\bar{X}_{n+1} \sim$ Cauchy under the assumption $\bar{X}_n \sim$ Cauchy. In order to do so, we rewrite

$$\bar{X}_{n+1} = \frac{n}{n+1}\bar{X}_n + \frac{1}{n+1}X_{n+1}.$$

As $\frac{n}{n+1}\bar{X}_n$ and $\frac{1}{n+1}X_{n+1}$ are obviously independent (see Exercise 5.2), we can convolute their densities to obtain the density of $\bar{X}_{n+1}$ as

$$f_{\bar{X}_{n+1}} = f_{\frac{n}{n+1}\bar{X}_n} \star f_{\frac{1}{n+1}X_{n+1}}$$

To calculate this convolution, we can use the above note and the hint by denoting $a := \frac{n}{n+1}$ and $b := \frac{1}{n+1}$ to obtain, using $a + b = 1$, that

$$\begin{aligned}
f_{\bar{X}_{n+1}}(x) &= \int_{-\infty}^{+\infty} f_{\frac{n}{n+1}\bar{X}_n}(y) f_{\frac{1}{n+1}X_{n+1}}(x-y)dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{\pi}\frac{a}{a^2+y^2}\frac{1}{\pi}\frac{b}{b^2+(x-y)^2}dy \\
&= \frac{ab}{\pi^2}\int_{-\infty}^{+\infty} \frac{1}{(a^2+y^2)(b^2+(x-y)^2)}dy \\
&= \frac{ab}{\pi^2}\frac{\pi(a+b)}{ab\left((a+b)^2+x^2\right)} \\
&= \frac{1}{\pi}\frac{(a+b)}{\left((a+b)^2+x^2\right)} \\
&= \frac{1}{\pi}\frac{1}{1+x^2}.
\end{aligned}$$

(b) The characteristic function of a Cauchy-distributed random variable is given by $\varphi_{X_i}(u) = e^{-|u|}$. We define $S_n := \sum_{i=1}^n X_i$ and use Lemma 4.2.1 to get

$$\varphi_{S_n}(u) = \prod_{i=1}^n \varphi_{X_i}(u)$$

and then

$$\varphi_{\bar{X}_n}(u) = \mathbb{E}\left[e^{iu\frac{1}{n}S_n}\right] = \varphi_{S_n}\left(\frac{1}{n}u\right) = \prod_{i=1}^n \varphi_{X_i}\left(\frac{u}{n}\right) = \prod_{i=1}^n e^{-\left|\frac{u}{n}\right|} = e^{-\sum_{i=1}^n \left|\frac{u}{n}\right|} = e^{-n\left|\frac{u}{n}\right|} = e^{-|u|}.$$

This is equivalent to $\bar{X}_n$ being Cauchy-distributed (because of Satz 4.2.3).

(c) The result does neither contradict the weak law of large numbers nor the strong law of large numbers nor the central limit theorem, since all these theorems require $\mathbb{E}[|X_i|]$ or even $\mathbb{E}\left[|X_i|^2\right]$ to be finite, but in the case of the Cauchy distribution, $\mathbb{E}[|X_i|] = \infty = \mathbb{E}\left[|X_i|^2\right]$.

Exercise 9.2 We should like to compute

$$A := \int_{-3}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

by using the Monte Carlo method:

(a) Express A in the form $\mathbb{E}[f(X)]$, where X is a standard normal random variable and f an appropriate function.

(b) Take an i.i.d. family $(X_i)_{i\in\mathbb{N}}$ having the same distribution as X. Set

$$A_n := \frac{1}{n}\sum_{i=1}^n f(X_i).$$

What is the distribution of $A_n - A$?

(c) Compute $\mathbb{E}[A_n]$ and show that $\operatorname{Var}[A_n] = (A - A^2)/n$.

(d) Show that for any $x > 0$, $\mathbb{P}[|A_n - A| \geq x] \leq 1/(4nx^2)$. This means that $A_n - A$ converges to 0 in probability when $n \to \infty$.

(e) Which theorem can you apply to get directly the above convergence?

(f) Does $A_n - A$ converge to 0 almost surely too?

Solution 9.2

(a) Set $f = I_{[-3,1]}$. Then

$$A = \mathbb{E}[f(X)] = \mathbb{P}[f(X) = 1] = \mathbb{P}[X \in [-3, 1]].$$

(b) nA_n has a binomial distribution with parameters n and A. Hence

$$\mathbb{P}[nA_n = k] = \binom{n}{k} A^k (1-A)^{n-k},$$

or equivalently

$$\mathbb{P}[A_n - A = k/n - A] = \binom{n}{k} A^k (1-A)^{n-k}.$$

(c) Because $nA_n \sim \mathrm{Bin}(n, A)$, we have $\mathbb{E}[A_n] = \frac{1}{n}nA = A$ and $\mathrm{Var}[A_n] = \frac{1}{n^2}nA(1-A) = (A - A^2)/n$. Alternatively, because the X_i are i.i.d., we have $\mathbb{E}[A_n] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}[f(X_i)] = \mathbb{E}[f(X)] = A$ and $\mathrm{Var}[A_n] = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}[f(X_i)] = \frac{\mathrm{Var}[f(X)]}{n} = (A - A^2)/n$, using that $f(X) = f^2(X)$ so that $\mathrm{Var}[f(X)] = \mathbb{E}[f(X)] - (\mathbb{E}[f(X)])^2$

(d) By the Chebyshev inequality, because $\mathbb{E}[A_n] = A$,

$$\mathbb{P}[|A_n - A| \geq x] \leq \frac{\mathrm{Var}[A_n]}{x^2} \leq \frac{1}{4nx^2},$$

since $A - A^2 \leq \frac{1}{4}$. The convergence is valid for all $x > 0$, which means that A_n converges in probability to A. To numerically approximate the value A, one can sample independently a family $(X_i)_{i=1\ldots n}$ having the standard normal distribution, count the number of points in the interval $[-3, 1]$ and then divide by n. As n becomes larger, we get a better approximation of A.

(e) We can apply the weak law of large numbers.

(f) Yes, because of the strong law of large numbers.

Exercise 9.3 Compute $\lim_{n\to\infty} e^{-n}\sum_{k=0}^n \frac{n^k}{k!}$.
Hint: You can use the central limit theorem for i.d.d. random variables $(X_i)_{i\in\mathbb{N}}$ such that $X_i \sim \mathrm{Poisson}(1)$.

Solution 9.3 If we define $S_n := \sum_{i=1}^n X_i$, then $S_n \sim \mathrm{Poisson}(n)$ and we have that

$$e^{-n}\sum_{k=0}^n \frac{n^k}{k!} = \mathbb{P}[S_n \leq n] = \mathbb{P}[S_n \leq n\mathbb{E}[X_1]] = \mathbb{P}\left[\frac{1}{\sqrt{n}}(S_n - n\mathbb{E}[X_1]) \leq 0\right] \to \Phi(0) = \frac{1}{2}$$

by the central limit theorem.

Exercise 9.4 Let X and Y be two independent standard normal random variables. Define the random variable

$$Z := X\,\mathrm{sign}(Y) := \begin{cases} X & \text{if } Y \geq 0, \\ -X & \text{if } Y < 0. \end{cases}$$

(a) Compute the distribution of Z.

(b) Compute the correlation between X and Z.

(c) Compute $\mathbb{P}[X + Z = 0]$.

(d) Does (X, Z) follow a multivariate normal distribution (in other words, is (X, Z) a Gaussian vector)?

Solution 9.4

(a) We compute, by independence of X and Y,

$$\begin{aligned}\mathbb{P}[Z \geq t] &= \mathbb{P}[\{X \geq t, Y \geq 0\} \cup \{X \leq -t, Y < 0\}] \\ &= \frac{1}{2}\mathbb{P}[X \geq t] + \frac{1}{2}\mathbb{P}[X \leq -t] \\ &= \mathbb{P}[X \geq t].\end{aligned}$$

Therefore, Z has the same law as X, thus $Z \sim \mathcal{N}(0, 1)$.

(b) Using the definition of covariance plus independence of X and Y gives

$$\begin{aligned}\operatorname{Cov}(X,Z) &= \mathbb{E}[XZ] - \mathbb{E}[X]\,\mathbb{E}[Z] \\ &= \mathbb{E}\left[X^2 I_{\{y\geq 0\}}\right] + \mathbb{E}\left[-X^2 I_{\{y<0\}}\right] = \frac{1}{2}\mathbb{E}\left[X^2\right] - \frac{1}{2}\mathbb{E}\left[X^2\right] = 0.\end{aligned}$$

Hence, the correlation is zero.

(c) We have again by independence of X and Y that

$$\mathbb{P}[X+Z=0] = \mathbb{P}[Y<0] + \mathbb{P}[Y\geq 0, 2X=0] = \frac{1}{2}.$$

(d) The vector (X,Z) is not a multivariate normal, because the sum of the two coordinates is not normally distributed, as it would have to be by Example 2.4.7.

Exercise 9.5 (Repetition of Exercise 5.3). Let $(S_n)_{n=0,1,\ldots,N}$ be a random walk and recall the family $\mathcal{F}_n$ of events observable up to time n. Every $A \in \mathcal{F}_n$ can be written as a union of sets from a partition $\mathcal{G}_n$ of Ω, and we define

$$\mathbb{E}[Z \mid \mathcal{F}_n] := \mathbb{E}[Z \mid \mathcal{G}_n]$$

for any random variable Z. Let $Y_n = \exp\left(S_n/\sqrt{N}\right)$ for $n = 0,1,\ldots,N$ and define $Z_n := \mathbb{E}[Y_N \mid \mathcal{F}_n]$ for $n = 0,1,\ldots,N$.

(a) Show that $Z_n := Y_n \left(\cosh\left(1/\sqrt{N}\right)\right)^{N-n}$ for $n = 0,1,\ldots,N$.

(b) Prove that $\mathbb{E}[Z_n \mid \mathcal{F}_m] = Z_m$ for $m \leq n$. (This means that Z is a martingale.)

(c) Prove that $\mathbb{E}\left[S_N^2 | \mathcal{F}_n\right] = S_n^2 + N - n$ for $n = 0,1,\ldots,N$

Hint: Show first that $S_N - S_n$ is independent of $\mathcal{G}_n$. You can use without proof the fact that $X_1, X_2, \ldots, X_N$ are i.i.d. for every $N \in \mathbb{N}$, that $\mathcal{F}_n$ is the σ-algebra generated by $X_1, X_2, \ldots, X_n$ and hence the independence of $(\mathcal{F}_n, X_{n+1}, X_{n+2}, \ldots, X_N)$. Use Exercises 5.1 and 5.2.
Solution 9.5

(a) As $\mathcal{G}_n \subseteq \mathcal{F}_n$, $(\mathcal{G}_n, X_{n+1}, X_{n+2}, \ldots, X_N)$ is independent as well. This can be used together with Exercise 5.2 to deduce the hint and show that $S_N - S_n$ is independent of $\mathcal{G}_n$. Now, we can proceed with the help of Exercise 5.1 (1) to get

$$\begin{aligned}
Z_n(\omega) &= \mathbb{E}\left[\exp\left(\frac{S_N}{\sqrt{N}}\right)\middle|\mathcal{G}_n\right](\omega) \\
&= \mathbb{E}\left[\exp\left(\frac{S_N - S_n + S_n}{\sqrt{N}}\right)\middle|\mathcal{G}_n\right](\omega) \\
&= \mathbb{E}\left[\exp\left(\frac{S_N - S_n + s}{\sqrt{N}}\right)\right]\Big|_{s=S_n(\omega)} \\
&= \exp\left(\frac{s}{\sqrt{N}}\right)\Big|_{s=S_n(\omega)} \mathbb{E}\left[\exp\left(\frac{S_N - S_n}{\sqrt{N}}\right)\right] \\
&= \exp\left(\frac{S_n(\omega)}{\sqrt{N}}\right)\mathbb{E}\left[\prod_{k=n+1}^{N}\exp\left(\frac{X_k}{\sqrt{N}}\right)\right] \\
&= Y_n(\omega)\prod_{k=n+1}^{N}\mathbb{E}\left[\exp\left(\frac{X_k}{\sqrt{N}}\right)\right] \\
&= Y_n(\omega)\prod_{k=n+1}^{N}\frac{1}{2}\left(\exp\left(\frac{-1}{\sqrt{N}}\right) + \exp\left(\frac{+1}{\sqrt{N}}\right)\right) \\
&= Y_n(\omega)\prod_{k=n+1}^{N}\cosh\left(1/\sqrt{N}\right) \\
&= Y_n(\omega)\left(\cosh\left(1/\sqrt{N}\right)\right)^{N-n}.
\end{aligned}$$

(b) First note that the vector space of all $\mathcal{G}_m$-measurable random variables is a subspace of the space of all $\mathcal{G}_n$-measurable functions since $\mathcal{G}_m \subseteq \mathcal{G}_n$. Since the concatenation of orthogonal projections on subspaces $\mathcal{G}_m, \mathcal{G}_n$ with $\mathcal{G}_m \subseteq \mathcal{G}_n$ is equal to the orthogonal projection on the smaller subspace $\mathcal{G}_m$, we obtain

$$\mathbb{E}\left[Z_n \mid \mathcal{F}_m\right] = \mathbb{E}\left[\mathbb{E}\left[Y_N \mid \mathcal{G}_n\right] \middle| \mathcal{G}_m\right] = \mathbb{E}\left[Y_N \mid \mathcal{F}_m\right] = Z_m.$$

Alternatively, the same argument as in (a) gives for $m < n$ that

$$\mathbb{E}\left[Y_n \mid \mathcal{F}_m\right] = \mathbb{E}\left[Y_n \mid \mathcal{G}_m\right] = Y_m\left(\cosh\left(1/\sqrt{N}\right)\right)^{n-m} = Z_m\left(\cosh\left(1/\sqrt{N}\right)\right)^{-(N-n)}$$

and so

$$\begin{aligned}
\mathbb{E}\left[Z_n \mid \mathcal{F}_m\right] &= \mathbb{E}\left[Y_n\left(\cosh\left(1/\sqrt{N}\right)\right)^{(N-n)}\middle|\mathcal{F}_m\right] \\
&= \left(\cosh\left(1/\sqrt{N}\right)\right)^{(N-n)}\mathbb{E}\left[Y_n \mid \mathcal{F}_m\right] = Z_m.
\end{aligned}$$

(c) Write $S_N^2 = (S_N - S_n + S_n)^2 = S_n^2 + 2S_n(S_N - S_n) + (S_N - S_n)^2$. Now $S_N - S_n$ is independent of $\mathcal{F}_n$, and S_n is $\mathcal{F}_n$-measurable. So by Exercise 5.1,

$$\begin{aligned}
&\mathbb{E}\left[S_n^2|\mathcal{F}_n\right] = S_n^2, \\
&\mathbb{E}\left[S_n(S_N - S_n)|\mathcal{F}_n\right] = S_n\mathbb{E}\left[S_N - S_n\right], \\
&\mathbb{E}\left[(S_N - S_n)^2|\mathcal{F}_n\right] = \mathbb{E}\left[(S_N - S_n)^2\right],
\end{aligned}$$

and because $S_N - S_n = \sum_{i=n+1}^{N} X_i$ with the X_i i.i.d. and $\mathbb{E}[X_i]$=0, $X_i^2 = 1$ so that $\mathrm{Var}[X_i] = 1$, we get

$$\mathbb{E}[S_N - S_n] = 0,$$
$$\mathbb{E}[(S_N - S_n)] = \mathrm{Var}[S_N - S_n] = \sum_{i=n+1}^{N} \mathrm{Var}[X_i] = N - n.$$

Putting all together gives

$$\mathbb{E}\left[S_N^2 \middle| \mathcal{F}_n\right] = S_n^2 + N - n$$

or equivalently

$$\mathbb{E}\left[S_N^2 - N \middle| \mathcal{F}_n\right] = S_n^2 - n,$$

which means that $(S_n^2 - n)_{n=0,1,\dots,N}$ is a martingale.

If you have feedback regarding the exercise sheets, please send a mail to Jakob Heiss.