Algebraic Topology II FS 2020

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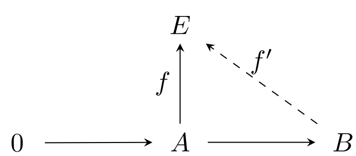
Exercise Sheet 1

Hom and Ext

1. Let G be an abelian group.
   (a) Let
   $$A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
   be a short exact sequence. Show that applying $\mathrm{Hom}(\,\cdot\,, G)$ yields an exact sequence
   $$0 \to C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^*,$$
   where $A^* = \mathrm{Hom}(A, G)$ etc.
   (b) Show that $\mathrm{Hom}(\mathbb{Z}, G) \cong G, \mathrm{Hom}(\mathbb{Z}^n, G) \cong G^n = \bigoplus_{i=1}^n G$ and $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

2. An abelian group A is called *divisible* if for any $a \in A$ and any nonzero $n \in \mathbb{Z}$ there exists an $a' \in A$ such that $a = na'$. A basic example is $\mathbb{Q}$ (with addition). Show that for any abelian group H and any divisible group G the group $\mathrm{Ext}(H, G) = \mathrm{Ext}_{\mathbb{Z}}(H, G)$ vanishes.

   Hint. *First show that any divisible group E is an injective object in the category of abelian groups, i.e. that for any exact sequence*
   $$0 \to A \to B$$
   of abelian groups and any homomorphism $f\colon A \to E$, there exists an extension $f'\colon B \to E$ such that the diagram

   commutes.

3. (a) Compute $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})$, $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ and $\mathrm{Ext}(\mathbb{Z}^n, \mathbb{Z})$.
   (b) Compute $\mathrm{Ext}(\mathbb{Z}, \mathbb{Q})$, $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q})$, $\mathrm{Ext}(\mathbb{Q}, \mathbb{Q})$.
   (c) Compute $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z}/m\mathbb{Z})$.

4. Let G, G', H, H' be abelian groups.

   In class, you have seen that for any free resolutions F and F' of H and H', respectively, a homomorphism $\alpha\colon H \to H'$ induces homomrphisms in cohomology $\alpha^*\colon H^n(F'; G) \to H^n(F; G)$. In particular, there is an induced map $\alpha^*\colon \mathrm{Ext}(H', G) = H^1(F'; G) \to \mathrm{Ext}(H; G) = H^1(F; G)$.
   (a) Show that for any free resolution
   $$F\colon \quad 0 \to F_1 \to F_0 \to H \to 0$$
   of H and any homomorphism $\beta\colon G \to G'$ there is a map $\beta_*\colon \mathrm{Ext}(H, G) \to \mathrm{Ext}(H, G')$ induced by the natural map $\beta_*\colon \mathrm{Hom}(F_1, G) \to \mathrm{Hom}(F_1, G'), \varphi \mapsto \beta \circ \varphi$.
   (b) Show that the maps $G \xrightarrow{n} G$, $H \xrightarrow{n} H$ multiplying each element by the integer n induce multiplication by n on $\mathrm{Ext}(H, G)$.

5. Let X be a topological space. Without using the universal coefficient theorem for cohomology show that the group $H^1(X; \mathbb{Z})$ has no torsion, i.e. no nontrivial element of finite order.