Algebraic Topology II FS 2020
Prof. Dr. Alessandro Sisto ETH Zürich
Paula Truöl

Exercise Sheet 2

Universal coefficient theorem and cup products

1. Compute the singular cohomology groups with $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ coefficients of the following spaces via simplicial or cellular cohomology and check the universal coefficient theorem in each case.
   (a) The two-dimensional torus $T = S^1 \times S^1$
   (b) The Klein bottle K
   (c) The real projective plane $\mathbb{R}P^2$

2. For an abelian group G, show that if $f\colon S^n \to S^n$ ($n \geq 1$) has degree d then $f^*\colon H^n(S^n;G) \to H^n(S^n;G)$ is multiplication by d.

3. Consider the *Moore space* $M(\mathbb{Z}/m\mathbb{Z}, n)$, which is obtained from the sphere S^n by attaching a single $(n+1)$-cell via an attaching map $S^n \to S^n$ of degree m.
   (a) For an abelian group G, compute $H_*(M(\mathbb{Z}/m\mathbb{Z}, n)) = H_*(M(\mathbb{Z}/m\mathbb{Z}, n);\mathbb{Z})$ and $H^*(M(\mathbb{Z}/m\mathbb{Z}, n);G)$.
   (b) Show that the quotient map $M(\mathbb{Z}/m\mathbb{Z}, n) \to M(\mathbb{Z}/m\mathbb{Z}, n)/S^n \cong S^{n+1}$, which contracts the S^n we started with to a point, induces trivial maps on the reduced homology $\widetilde{H}_i(M(\mathbb{Z}/m\mathbb{Z}, n);\mathbb{Z})$ for all i, but not on $H^{n+1}(M(\mathbb{Z}/m\mathbb{Z}, n);\mathbb{Z})$.
   (c) Conclude that the splitting in the universal coefficient theorem cannot be natural, i.e. there does not exist an isomorphism

$$i_X : H^n(X;\mathbb{Z}) \to \mathrm{Ext}(H_{n-1}(X),\mathbb{Z}) \oplus \mathrm{Hom}(H_n(X),\mathbb{Z})$$

   for every topological space X such that for every continuous map $f\colon X \to Y$ between topological spaces there is a commutative diagram

$$\begin{array}{ccccc}
H^n(X;\mathbb{Z}) & \xrightarrow[i_X]{} & \mathrm{Ext}(H_{n-1}(X),\mathbb{Z}) & \oplus & \mathrm{Hom}(H_n(X),\mathbb{Z}) \\
{\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle (f_*)^*} & & \uparrow{\scriptstyle (f_*)^*} \\
H^n(Y;\mathbb{Z}) & \xrightarrow[i_Y]{} & \mathrm{Ext}(H_{n-1}(Y),\mathbb{Z}) & \oplus & \mathrm{Hom}(H_n(Y),\mathbb{Z}).
\end{array}$$

4. For $G \in \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$, compute the cup product structure in $H^*(K;G)$ for the Klein bottle K. You can use pictures to define bases for H^* as in the examples in the lecture.