Algebraic Topology II FS 2020
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Exercise Sheet 3

Tensor products, cohomology rings, Künneth formula

1. Let R be a commutative ring and let M be an R-module. A homomorphism $f\colon A \to B$ of R-modules induces a group homomorphism $\mathrm{Id} \otimes_R f\colon M \otimes_R A \to M \otimes_R B$ given by $(\mathrm{Id} \otimes_R f)(m \otimes_R a) = m \otimes_R f(a)$.

   (a) Show that for every exact sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence
   $$M \otimes_R A \xrightarrow{\mathrm{Id} \otimes_R f} M \otimes_R B \xrightarrow{\mathrm{Id} \otimes_R g} M \otimes_R C \to 0$$
   is exact. *Hint:* To prove exactness at $M \otimes_R B$, construct an inverse for an appropriate map
   $$(M \otimes_R B)/\operatorname{Im}(\mathrm{Id} \otimes_R f) \to M \otimes_R C.$$

   (b) Show that free R-modules are flat, i.e. show that if M is a free R-module, then for every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence
   $$0 \to M \otimes_R A \xrightarrow{\mathrm{Id} \otimes_R f} M \otimes_R B \xrightarrow{\mathrm{Id} \otimes_R g} M \otimes_R C \to 0$$
   is exact.

2. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

3. Show that any continuous map $f\colon S^{k+l} \to S^k \times S^l$ induces the zero homomorphism
   $$H^{k+l}(S^k \times S^l; \mathbb{Z}) \to H^{k+l}(S^{k+l}; \mathbb{Z}) \qquad \text{for } k, l > 0.$$

4. Naturality of the cross product with respect to maps between spaces is immediate from the naturality of cup products. Show that the cross product is also natural with respect to coboundary maps in long exact sequences, i.e. for all pairs (X, A) and spaces Y, there is a commutative diagram
   $$\begin{array}{ccc} H^k(A;R) \otimes_R H^l(Y;R) & \xrightarrow{\delta \otimes_R \mathrm{Id}} & H^{k+1}(X,A;R) \otimes_R H^l(Y;R) \\ \downarrow \times & & \downarrow \times \\ H^{k+l}(A \times Y;R) & \xrightarrow{\delta} & H^{k+l+1}(X \times Y, A \times Y;R). \end{array}$$

5. Let $I = [0,1]$ and let Y be any space such that $H^{n-1}(Y;R)$ is a finitely generated free R-module for each n. Show that the cross product
   $$H^1(I, \partial I; R) \otimes_R H^{n-1}(Y;R) \xrightarrow{\times} H^n(I \times Y, \partial I \times Y; R)$$
   is an isomorphism. *Hint:* Consider the long exact sequence of the pair $(I \times Y, \partial I \times Y)$ and use the result of the previous problem.
   Remark: This is the special case of the relative Künneth formula we needed to compute the cohomology rings of $\mathbb{R}P^n$.