Algebraic Topology II FS 2020
Prof. Dr. Alessandro Sisto ETH Zürich
Paula Truöl

Exercise Sheet 5

Cohomology with compact support, Poincaré duality

1. Let X be a topological space and R a ring with identity. Show that $(\alpha \frown \varphi) \frown \psi = \alpha \frown (\varphi \smile \psi)$ for all $\alpha \in C_k(X;R), \varphi \in C^l(X;R)$ and $\psi \in C^m(X;R)$. Deduce that the cap product makes $H_*(X;R)$ a right $H^*(X;R)$-module.

2. Show that $H^0_c(X;G) = 0$ for any abelian group G if X is path-connected and non-compact.

3. For closed connected n-dimensional manifolds M_1 and M_2, their *connected sum* $M_1 \# M_2$ is the n-dimensional manifold obtained by deleting the interiors of closed n-balls $B_1 \subset M_1$ and $B_2 \subset M_2$ and identifying the resulting boundary spheres ∂B_1 and ∂B_2 via some homeomorphism between them. (Assume that each B_i embeds nicely in a larger ball in M_i.) Show that there are isomorphisms $H_i(M_1 \# M_2;\mathbb{Z}) \cong H_i(M_1;\mathbb{Z}) \oplus H_i(M_2;\mathbb{Z})$ for $0 < i < n$, with one exception: If both M_1 and M_2 are nonorientable, then $H_{n-1}(M_1 \# M_2)$ is obtained from $H_{n-1}(M_1;\mathbb{Z}) \oplus H_{n-1}(M_2;\mathbb{Z})$ by replacing one of the two $\mathbb{Z}/2\mathbb{Z}$-summands by a $\mathbb{Z}$-summand. *Hint:* Euler characteristics and the fact about torsion subgroups below may help in the exceptional case.

In the following, let M be a closed connected n-dimensional manifold. Then the homology groups of M are finitely generated and by the fundamental theorem of finitely generated abelian groups we can write each of them as a direct sum of a free abelian group and a finite group, its torsion subgroup. It can be shown that the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}/2\mathbb{Z}$ if M is nonorientable.
Recall from class that if M is orientable and n is odd, then the Euler characteristic $\chi(M)$ is zero. This corollary of Poincaré duality is also true without the orientability assumption on M (see Corollary 3.37 in [Hatcher]).

4. Let $n = 3$.
   (a) Write $H_1(M;\mathbb{Z})$ as $\mathbb{Z}^r \oplus F$, the direct sum of a free abelian group of rank r and a finite group F. Show that $H_2(M;\mathbb{Z})$ is $\mathbb{Z}^r$ if M is orientable and $\mathbb{Z}^{r-1} \oplus \mathbb{Z}/2\mathbb{Z}$ if M is nonorientable. In particular, $r \geq 1$ when M is nonorientable.
   (b) Show that if M is nonorientable, then the fundamental group $\pi_1(M)$ is infinite.
   (c) Show that if M is simply-connected, then $H_i(M;\mathbb{Z}) \cong H_i(S^3;\mathbb{Z})$ for all i.

5. Show that if M is orientable, $n = 2k$ and $H_{k-1}(M;\mathbb{Z})$ torsionfree, then $H_k(M;\mathbb{Z})$ is also torsionfree.