THM: · $H^*(\mathbb{R}P^m, \mathbb{Z}/_2) \cong \mathbb{Z}/_2[\alpha]/(\alpha^{m+1})$, $|\alpha|=1$

· $H^*(\mathbb{C}P^m, \mathbb{Z}) \cong \mathbb{Z}[\alpha]$, $|\alpha|=2$

TOOL WE NEED: EVEN MORE RELATIVE CUP PRODUCT

LET:
· X CW-CPLX, $A, B \subseteq X$ SUB-CPLXS, OR
· X TOP SPACE, $A, B \subseteq X$ OPEN

THEN $\cup: H^k(X,A;R) \times H^\ell(X,B;R) \to H^{k+\ell}(X, A\cup B;R)$
IS WELL-DEFINED

SUBTLETY: $\varphi \in C^k(X,A;R)$, $\psi \in C^\ell(X,B;R)$
$\Rightarrow \varphi|_{C_k(A)}=0$, $\psi|_{C_k(B)}=0$, SO
$\varphi\cup\psi \in C^{k+\ell}(X, A+B;R) =$
$\{\varphi \in C^{k+\ell} : \varphi|_{C_k(A)}=0, \varphi|_{C_k(B)}=0\}$
USUALLY STRICT! $\longrightarrow$ $\cup|$
BUT $C^{k+\ell}(X, A\cup B;R) = \{\varphi \in C^{k+\ell} : \varphi|_{C_k(A\cup B)}=0\}$

HOWEVER, IN THE CASES ABOVE, THE INCLUSIONS
$C^m(X, A\cup B;R) \hookrightarrow C^m(X, A+B;R)$ INDUCE
ISO IN COHOMOLOGY, BY PROOF OF EXCISION.

$\exists$ CROSS PRODUCT AND RELATIVE KÜNNETH FORMULA:

THM: $H^*(X,A;R) \otimes_R H^*(Y,B;R) \xrightarrow{\times} H^*(X\times Y, A\times Y \cup X\times B;R)$
IS ISO IF · (X,A), (Y,B) ARE CW-PAIRS
· $H^k(Y,B;R)$ IS FIN GEN FREE R-MODULE $\forall k$

EXAMPLES:
· $I=[0,1]$, $H^i(I^i, \partial I^i;R) \otimes_R H^j(I^j, \partial I^j;R) \xrightarrow{\times} H^{i+j}(I^{i+j}, \partial I^{i+j};R)$
IS ISO

NOTE: THIS FOR $i=1$ WOULD SUFFICE FOR THM

· $H^i(\mathbb{R}^i, \mathbb{R}^i-\{0\};R) \otimes_R H^j(\mathbb{R}^j, \mathbb{R}^j-\{0\};R)$
$\xrightarrow{\times} H^{i+j}(\mathbb{R}^{i+j}, \mathbb{R}^{i+j}-\{0\};R)$
IS ISO

NOW WE PROVE: $H^*(\mathbb{R}P^m; \mathbb{Z}/_2) \cong \mathbb{Z}/_2[\alpha]/(\alpha^{m+1})$, $|\alpha|=1$

FOR $\mathbb{C}P^m$: BASICALLY SAME PROOF, REPLACE ALL H^k WITH H^{2k}, AND $\mathbb{R}$ BY $\mathbb{C}$

NOTATION FOR PROOF: · $P^k = \mathbb{R}P^k$
· ALL H^* ARE WITH $\mathbb{Z}/_2$ COEFF.

RECALL: $H^i(P^m) = \begin{cases} \mathbb{Z}/_2 & i\leq m \leftarrow \alpha_i = \text{GENERATOR} \\ 0 & \text{OTHERWISE} \end{cases}$

$\mathbb{Z}/_2[\alpha]/(\alpha^{m+1}) \leftrightarrow c_0 + \dots + c_m\alpha^m$

NEED TO SHOW: $\alpha_i \cup \alpha_j = \alpha_{i+j}$

TRUE WITH $\mathbb{Z}/_2$ COEFF!

FROM CW-STRUCTURE ON P^m:
$P^{m-1} \hookrightarrow P^m$ INDUCES ISO ON H^i FOR $i\leq m-1$

HENCE: WE CAN DO INDUCTION AND JUST SHOW
THAT $\alpha_i \cup \alpha_{m-i} = \alpha_m$

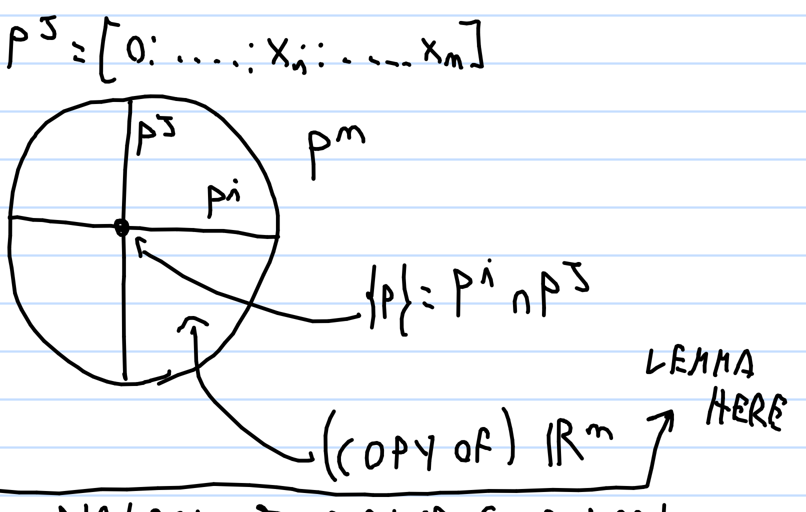
SET: $j = m-i$

$P^m = [x_0 : \dots : x_m]$

(COPY OF) $P^i = [x_0 : \dots : x_i : 0 \dots 0]$

" $P^j = [0 : \dots : x_i : \dots x_m]$

CARTOON:



THE RIGHT DIAGRAM TO REDUCE COMPUTATION TO EXAMPLE, ALL MAPS INDUCED BY INCLUSIONS

$$\begin{tikzcd}
H^i(P^m) \times H^j(P^m) \arrow[rr, "\cup"] & & H^m(P^m) \\
H^i(P^m, P^m-P^j) \times H^j(P^m, P^m-P^i) \arrow[u] \arrow[rr, "\cup"] & & H^m(P^m, P^m-\{p\}) \arrow[u] \\
H^i(\mathbb{R}^m, \mathbb{R}^m-\mathbb{R}^j) \times H^j(\mathbb{R}^m, \mathbb{R}^m-\mathbb{R}^i) \arrow[u, "\color{red}\bullet"] \arrow[d, "\color{blue}\bullet"] \arrow[rr, "\cup"] & & H^m(\mathbb{R}^m, \mathbb{R}^m-\{0\}) \arrow[u, "\color{red}\bullet"'] \\
H^i(\mathbb{R}^i, \mathbb{R}^i-\{0\}) \times H^j(\mathbb{R}^j, \mathbb{R}^j-\{0\}) \arrow[urr, "\times"] & &
\end{tikzcd}$$

GEN $\times$ GEN = GEN, BY EXAMPLE

CLAIM: ALL VERTICAL MAPS ARE ISO.

$\color{red}\bullet$ = ISO BY EXCISION

$\color{blue}\bullet$ = ISO BY HOMOTOPY EQUIV.

REMAINING ARROWS: FOLLOWS FROM LEMMA (UP TO CHANGING COORDINATES)

LEM: $H^i(P^m, P^m-P^j) \to H^i(P^m)$ INDUCED BY INCLUSION IS ISO

PROOF: WE USE FACTORISATION:

$H^i(P^m, P^m-P^j) \to H^i(P^m, P^{i-1}) \to H^i(P^m)$

SECOND ARROW IS ISO BY CELLULAR COHOM.

$P^m - P^j = \{[x_0 : \dots : x_m] \mid \exists k\leq i-1 \ x_k \neq 0\}$

HENCE, P^m-P^j DEFORMATION RETRACTS TO P_{i-1}.
VIA:
$([x_0 : \dots : x_m], t) \mapsto ([x_0 : \dots : x_{i-1} : tx_i : \dots : tx_m])$

HENCE: INCLUSION $(P^m, P^{i-1}) \hookrightarrow (P^m, P^m-P^j)$ IS
HOMOT EQUIV $\Rightarrow$ FIRST ARROW IS ISO $\square$

FUN FACTS: · $\exists$ QUATERNIONIC PROJECTIVE SPACES $\mathbb{H}P^m$, AND
$H^*(\mathbb{H}P^m; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{m+1})$, $|\alpha|=4$

· $H^*(\mathbb{R}P^{2k}; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$ $|\alpha|=2$

· $H^*(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}[\alpha,\beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta)$
$|\alpha|=2$ $|\beta|=2k+1$

CAN BE COMPUTED FROM $\mathbb{Z}/_2$ CASE BY CHANGE OF COEFFICIENT MAPS