RECALL: $\cdot\ H^*(\mathbb{RP}^{m-1};\mathbb{Z}/2) \overset{(1)}{\cong} \mathbb{Z}/2[\alpha]/(\alpha^m)$

$\cdot\ H^*(\mathbb{CP}^{m-1};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^m)$

$\cdot\ H^*(X\times Y;R) \cong H^*(X;R)\otimes_R H^*(Y;R)$

IF...

NEXT: TWO APPLICATIONS

GIVEN m, IS THERE A "NICE" MULTIPLICATION ON $\mathbb{R}^m$?

"NICE" = WITH INVERSES FOR NON-0 ELEMENTS

E.G. $\cdot\ m=1$ ✓

$\cdot\ m=2$ $\mathbb{R}^2 \cong \mathbb{C}$, SO YES

$\cdot\ m=4$ $\mathbb{R}^2 \cong \mathbb{H}$, QUATERNIONS, SO YES

$m=3$?

THM 1: LET m BE S.T. $\exists$ $\mathbb{R}$-BILINEAR MAP

$\mu: \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ S.T.

$\mu(x,y)=0 \Leftrightarrow x=0$ OR $y=0$.

THEN m IS A POWER OF 2

HARDER THM: SAME HYPOTHESES $\Rightarrow m \in \{1,2,4,8\}$

OCTONIONS

NOTE: KNOWN PROOFS OF HARDER THM ALSO USE ALGEBRAIC TOPOLOGY

WE WILL USE:

EXERCISE: $H^*(\mathbb{RP}^{m-1}\times\mathbb{RP}^{m-1};\mathbb{Z}/2) \overset{(2)}{\cong} \mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_1^m,\alpha_2^m)$

MOREOVER, LET $i: \mathbb{RP}^{m-1} \longrightarrow \mathbb{RP}^{m-1}\times\mathbb{RP}^{m-1}$

$q \longmapsto (p, q)$

UNDER THE ISO (1) AND (2), WE HAVE

$i^*: \mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_i^m) \longrightarrow \mathbb{Z}/2[\alpha]/(\alpha^m)$

SATISFIES: $i^*(\alpha_1)=\alpha$, $i^*(\alpha_2)=0$

HINT: USE DESCRIPTION OF $H^*(\mathbb{RP}^{m-1};\mathbb{Z}/2)$ + KÜNNETH □

USE DESCR OF ISO!

PROOF OF THM 1:

NOTE: μ IS CONTINUOUS (BILINEAR MAPS ARE CONT.)

μ INDUCES A WELL-DEF, CONTINUOUS MAP

$\bar{\mu}: \mathbb{RP}^{m-1}\times\mathbb{RP}^{m-1} \longrightarrow \mathbb{RP}^{m-1}$

MOREOVER, $\forall\ p\in\mathbb{RP}^{m-1}$ WE HAVE:

$\cdot\ \bar{\mu}|_{\mathbb{RP}^{m-1}\times\{p\}}$ IS HOMEO ONTO $\mathbb{RP}^{m-1}$

$\cdot$ SAME FOR $\{p\}\times\mathbb{RP}^{m-1}$

WHAT DOES

$\bar{\mu}^*: \underset{\cong\ \mathbb{Z}/2[\alpha]/(\alpha^m)}{H^*(\mathbb{RP}^{m-1};\mathbb{Z}/2)} \longrightarrow \underset{\cong\ \mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_i^m)}{H^*(\mathbb{RP}^{m-1}\times\mathbb{RP}^{m-1};\mathbb{Z}/2)}$

LOOK LIKE?

$\bar{\mu}^*(\alpha) = \underline{k_1\alpha_2 + k_2\alpha_2}$ FOR SOME $k_1,k_2\in\mathbb{Z}/2$

$|\bar{\mu}^*(\alpha)|=1$, SO IT HAS TO HAVE THIS FORM

CLAIM: $k_1=k_2=1$

PROOF OF CLAIM:

SAY BY CONTRADICTION $k_1=0$. CONSIDER:

$\mathbb{RP}^{m-1} \xrightarrow{i} \mathbb{RP}^{m-1}\times\mathbb{RP}^{m-1} \xrightarrow{\bar{\mu}} \mathbb{RP}^{m-1}$

$q \longmapsto (q,p)$

AT THE LEVEL OF H^*:

$0 \xleftarrow{\ i^*\ } \cancel{0\cdot\alpha_1}+k_2\alpha_2 \xleftarrow{\ \bar{\mu}^*\ } \alpha$

BY EXERCISE

SO $i^*\circ\bar{\mu}^* = (\bar{\mu}\circ i)^*$ IS 0-MAP, WHICH CONTRADICTS THAT $\bar{\mu}\circ i$ IS HOMEO □

SINCE $\alpha^m=0$, WE MUST HAVE

$\bar{\mu}^*(\alpha^m) = (\alpha_1+\alpha_2)^m = 0 \in \mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_i^m)$

$(\alpha_1+\alpha_2)^m = \sum \binom{m}{k}\alpha_1^k\alpha_2^{m-k}$, SO WE HAVE:

$\binom{m}{k}=0\in\mathbb{Z}/2\ \ \forall\ 0<k<m \quad (*)_m$

FINAL CLAIM: $(*) \Rightarrow m$ IS A POWER OF 2

PROOF: NOTE THAT $(*)_m \Leftrightarrow$ IN $\mathbb{Z}/2[x]$

WE HAVE $(1+x)^m = 1+x^m$.

SUPPOSE THAT m IS NOT A POWER OF 2,

SO $m=2^i\cdot s$, $s>1$ ODD.

THEN, USING $(a+b)^2=a^2+b^2$, WE HAVE

$(1+x)^m = \left((1+x)^{2^i}\right)^s = \left(1+x^{2^i}\right)^s = 1+\underline{s\cdot x^{2^i}}+\dots+x^m$

$\neq 1+x^m$ □

WITH A SIMILAR STRATEGY USING $\mathbb{Z}$ COEFFICIENTS WE CAN PROVE

THM. LET m BE S.T. $\exists$ $\mathbb{C}$-BILINEAR MAP

$\mu: \mathbb{C}^m\times\mathbb{C}^m \longrightarrow \mathbb{C}^m$ S.T.

$\mu(x,y)=0 \Leftrightarrow x=0$ OR $y=0$.

THEN $m=1$

(PUNCHLINE: $\binom{m}{k}=0\in\mathbb{Z}\ \ \forall\ 0<k<m \Leftrightarrow m=1$)

EXERCISE: $H^*(\overset{X}{\overset{\|}{\mathbb{RP}^{m-1}}}\times\mathbb{RP}^{m-1};\mathbb{Z}/2) \overset{(2)}{\cong} \mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_1^m,\alpha_2^m)$

MOREOVER, LET $i: \mathbb{RP}^{m-1} \longrightarrow \mathbb{RP}^{m-1}\times\mathbb{RP}^{m-1}$

$q \longmapsto (q, p)$

UNDER THE ISO (1) AND (2), WE HAVE

$i^*: \mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_i^m) \longrightarrow \mathbb{Z}/2[\alpha]/(\alpha^m)$

SATISFIES: $i^*(\alpha_1)=\alpha$, $i^*(\alpha_2)=0$

SKETCH: ALL H^* HAVE $\mathbb{Z}/2$ COEFF

KÜNNETH GIVES ISO

$H^*(X)\otimes_{\mathbb{Z}/2} H^*(X) \xrightarrow{\ \times\ } H^*(X\times X)$

$a\otimes b \longmapsto p_1^*(a)\cup p_2^*(b)$

$p_1(x,y)=x$

$p_2(x,y)=y$

$H^*(X)\cong\mathbb{Z}/2[\alpha]/(\alpha^m)$

LET'S DENOTE α ALSO THE NON-TRIVIAL CLASS IN H_1

NON-TRIV ELEM OF H^0

THEN WE WANT $\alpha_1 = p_1^*(\alpha) = \times(\alpha\otimes 1)$

$\alpha_2 = p_2^*(\alpha) = \times(1\otimes\alpha)$

ALGEBRAIC FACT: THIS IS ISO:

$\mathbb{Z}/2[\alpha_1,\alpha_2]/(\alpha_i^m) \to \mathbb{Z}/2[\alpha_1]/(\alpha_1^m)\otimes_{\mathbb{Z}/2}\mathbb{Z}/2[\alpha_2]/(\alpha_2^m)$

$\alpha_1 \longmapsto \alpha_1\otimes 1$

$\alpha_2 \longmapsto 1\otimes\alpha_2$

THEN $i^*(\alpha_1) = i^*(p_1^*(\alpha)) = (p_1\circ i)^*(\alpha)$

$X \xrightarrow{i} X\times X \xrightarrow{p_1} X$

$q \longmapsto (q,p) \longmapsto q$

$\Rightarrow p_1\circ i = id \Rightarrow (p_1\circ i)^* = id \Rightarrow i^*(\alpha_1)=\alpha$ ✓

$i^*(\alpha_2)$, $p_2\circ i = $ CONST. $\Rightarrow (p_2\circ i)^*: H^1(X)$ ↺

IS 0-MAP $\Rightarrow i^*(\alpha_2)=0$ ✓