FOR WHICH k, m $\exists$ "INTERESTING" MAPS $S^k \to S^m$?

$k<m$: ALL CONT MAPS $S^k \to S^m$ ARE HOMOTOPIC TO A CONST MAP (E.G. $\pi_1(S^2) = \{e\}$)

↳ CAN BE PROVEN USING SIMPLICIAL APPROXIMATION.

$k=m$: INTERESTING INVARIANT: DEGREE

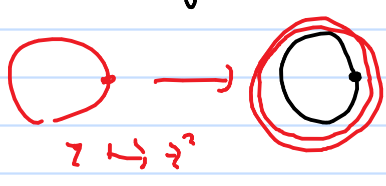
$f: S^m \to S^m$ CONT, THEN

$$f_*: \underset{\mathbb{Z}}{\underset{\text{|||}}{H_m(S^m;\mathbb{Z})}} \to \underset{\mathbb{Z}}{\underset{\text{|||}}{H_m(S^m;\mathbb{Z})}}$$

IS MULTIPLICATION BY $\deg(f)$, BY DEFINITION OF $\deg(f)$

- $\deg(\text{const}) = 0$, $\deg(id) = 1$, $\deg(\text{REFL}) = -1$,

EXAMPLE OF DEGREE 2:

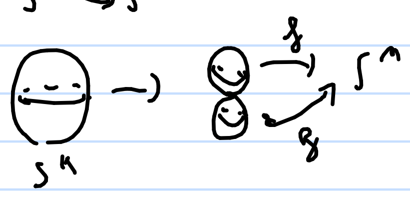


FACT: f^* IS ALSO MULTIPLICATION BY $\deg(f)$

$k>m$: IT VARIES A LOT

DEFN. <u>HOMOTOPY GP</u> $\pi_k(S^m)$ HAS ELEMENTS: HOMOT. CLASSES OF POINTED MAPS $S^k \to S^k$

OPERATION: fg =



E.G. $\pi_k(S^m) = \{e\}$ IF $k<m$

- THM: $\pi_m(S^m) \cong \mathbb{Z}$, ISOMORPHISM GIVEN BY DEGREE

<u>$\pi_k(S^m)$ TABLE</u>

HOPF MAP(S)

LET $h: S^3 \to S^2$ BE $h(x) = [x]$, where $S^3 \subseteq \mathbb{C}^2$ and $S^2 \cong \mathbb{C}P^1$

↳ <u>COMPLEX HOPF MAP</u>

CAN ALSO THINK: $x \mapsto$ COMPLEX LINE THROUGH x

NOTE: $\forall p \in \mathbb{C}P^1$, SAY $p = [x]$, $|x| = 1$, WE HAVE $h^{-1}(p) = \{wx : w \in S^1\} \cong S^1$

IN FACT, h^{-1}(NORTH/SOUTH HEMISPHERE) $\cong D^2 \times S^1$

SO S^3 DECOMPOSES AS A UNION OF SOLID TORI

THM: h IS NOT HOMOTOPIC TO CONST MAP

NOTE: ALL MAPS $S^3 \to S^2$ INDUCE SAME MAPS $H_i(S^3) \to H_i(S^2)$ AND $H^i(S^2;R) \to H^i(S^3;R)$!

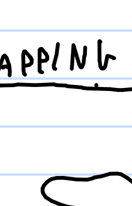
USEFUL CONSTRUCTIONS:

X TOP SPACE:

- <u>CONE OVER</u> X $CX = X \times [0,1] / X \times \{0\}$

← COPY OF X

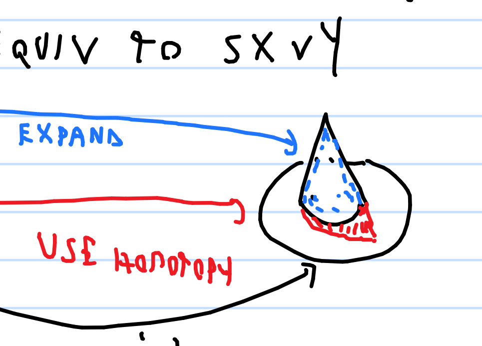
- <u>SUSPENSION OF X</u> $SX = X \times [0,1] / \{X \times \{0\}, X \times \{1\}\}$



E.G. - $CS^m \cong D^{m+1}$
- $S(S^m) \cong S^{m+1}$

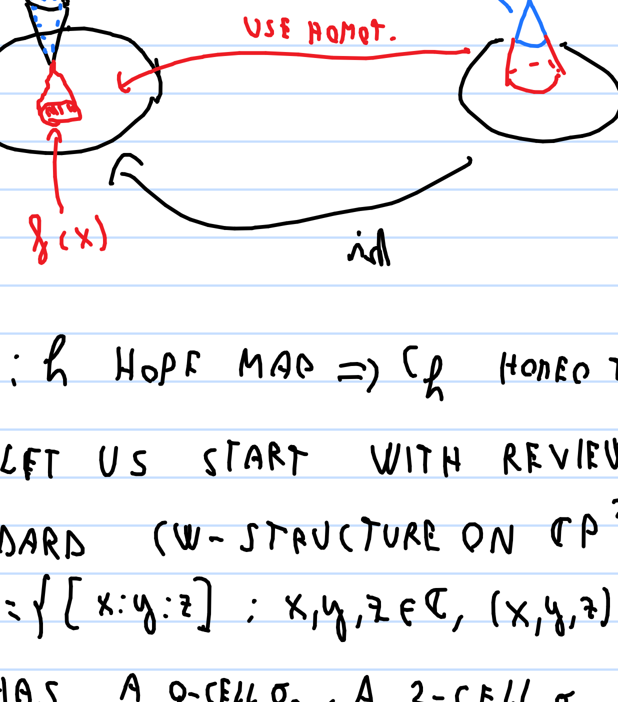
- $f: X \to Y$ <u>MAPPING CONE</u> $C_f = CX \cup_f Y = CX \sqcup Y / [(x,1)] \sim f(x)$

PICTURE:



LEMMA: f HOMOTOPIC TO CONST $\Rightarrow$ C_f IS HOMOTOPY EQUIV TO $SX \vee Y$

PICTURES:



LEMMA: h HOPF MAP $\Rightarrow$ C_h HOMEO TO $\mathbb{C}P^2$

PROOF: LET US START WITH REVIEWING THE STANDARD CW-STRUCTURE ON $\mathbb{C}P^2$

$\mathbb{C}P^2 = \{[x:y:z] : x,y,z \in \mathbb{C}, (x,y,z) \neq (0,0,0)\}$

$\mathbb{C}P^2$ HAS A 0-CELL σ_0, A 2-CELL σ_2, A 4-CELL σ_4

σ_0: $[0:0:1]$; σ_2: IMAGE $= \{[0:x:y]\} \cong \mathbb{C}P^1$

CHARACTERISTIC MAPS

$$f_{\sigma_2}: D^2 \to \mathbb{C}P^2$$
$$r \cdot v \mapsto [0 : \tfrac{1}{2} - 1 : v]$$
$r \in [0,1]$, $v \in S^1$

$$f_{\sigma_4}: D^4 \to \mathbb{C}P^2$$
$$r \cdot (x,y) \mapsto [\tfrac{1}{2} - 1 : x : y]$$
$r \in [0,1]$, $(x,y) \in S^3$

ATTACHING MAP $f_{\partial\sigma_4}: \partial D^4 = S^3 \to \mathbb{C}P^1 \subseteq \mathbb{C}P^2$, $x \mapsto [x]$

↳ IT'S h!

SUMMARY: h IS ATTACHING MAP OF THE 4-CELL OF $\mathbb{C}P^2$

SO, BACK TO C_h:

$C_h = CS^3 \cup_h \mathbb{C}P^1 \cong D^4 \cup_h \mathbb{C}P^1 \cong \mathbb{C}P^2$ □

RECALL: f HOMOTOPIC TO CONST $\Rightarrow$ $C_f \simeq SX \vee Y$

WE HAVE:

$H^*(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^3)$, $|\alpha| = 2$

$H^*(S^4 \vee S^2;\mathbb{Z}) \cong \mathbb{Z}[s,t]/(t^2, s^2)$, $|s| = 2$, $|t| = 4$ } NOT ISOMORPHIC

SO $C_h \not\simeq S(S^3) \vee S^2 \Rightarrow h \not\simeq$ CONST. (where $S(S^3) = S^4$)

THERE ARE ALSO:

- QUATERNIONIC HOPF MAP $S^7 \to S^4$
- OCTONIONIC " $S^{15} \to S^8$