Q: WHICH GRADED RINGS ARE COHOMOLOGY RINGS?

LET'S CONSIDER POLYNOMIAL RING $R[\alpha]$, WITH $|\alpha| = d$, R COMMUTATIVE

RMK: d MUST BE EVEN IF $1 \neq -1$ IN R: $\alpha^2 = (-1)^{d^2} \alpha^2$

- FOR $R = \mathbb{Z}$, $\cdot H^*(\mathbb{C}P^\infty, \mathbb{Z}) = \mathbb{Z}[\alpha]$, $|\alpha| = 2$
  $\cdot H^*(\mathbb{H}P^\infty, \mathbb{Z}) = \mathbb{Z}[\alpha]$, $|\alpha| = 4$

THM 1: OTHER DIMENSIONS FOR α ARE NOT POSSIBLE IF $R = \mathbb{Z}$

INSTEAD:

THM 2: $\forall$ d EVEN $\exists$ TOP SPACE X S.T.

$$H^*(X; \mathbb{Q}) \cong \mathbb{Q}[\alpha] \text{ WITH } |\alpha| = d$$

WE USE: JAMES REDUCED PRODUCT

X TOP SPACE, $p \in X$

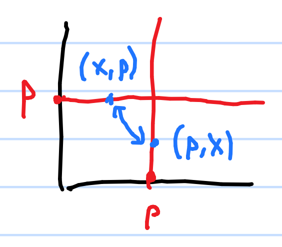
$$J(X) := \coprod_{k \geq 1} X^k \Big/ (x_1, \dots, x_k) \sim (x_1, \dots, \hat{x}_i, \dots x_k)$$

$J_m(X) :=$ IMAGE OF $\coprod_{1 \leq k \leq m} X^m$ IN $J(X)$.

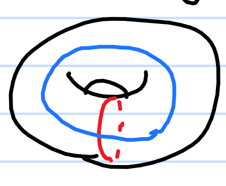
$J_m(X)$ IS NATURALLY HOMEO TO:

$$X^m \Big/ (x_1, \dots, x_i, p, \dots x_m) \sim (x_1, \dots, p, x_i, \dots x_m)$$

$m = 2$



$J_2(S^1)$



GLUE RED AND BLUE

LEMMA: X CW-CPLX, p 0-CELL $\Rightarrow$ $J(X)$ AND $J_m(X)$ ARE CW CPLXS

IDEA: THE EQUIV RELN ON J^m IDENTIFIES CELLS OF X^m WITH CELLS OF X^m.

THE FOLLOWING PROP IMPLIES THM 2:

PROP: LET d BE EVEN. THEN:

$$H^p(J(S^d); \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p \mid d \\ 0 & p \nmid d \end{cases}$$

. LET x_d BE A GENERATOR OF $H^d(J(S^d); \mathbb{Z})$

THEN, FOR ALL $m \geq 1$, x_d^m IS $m!$ TIMES A GENERATOR x_{dm} OF $H^{dm}(J(S^d); \mathbb{Z})$

PROP $\Rightarrow$ $H^*(J(S^d); \mathbb{Q}) \cong \mathbb{Q}[\alpha]$, $|\alpha| = d$ $\forall$ EVEN d

SKETCH OF PROP, FOR $d = 2$ (FOR OTHER d: SAME PROOF)

$J(S^2)$ HAS CW-STRUCTURE WITH ONE CELL IN EACH EVEN DIMENSION $\Rightarrow$ DESCRIPTION OF $H^p(J(S^2); \mathbb{Z})$

LET $q: (S^2)^m \longrightarrow J_m(S^2)$ BE QUOTIENT MAP.

IT'S A GOOD IDEA TO CONSIDER IT BECAUSE:

- $q^*: H^{2m}(J_m(S^2); \mathbb{Z}) \longrightarrow H^{2m}((S^2)^m; \mathbb{Z})$ IS ISO BY CELL COHOM (THERE ARE CELLS ONLY IN EVEN DIMENSIONS
- $J_m(S^2) \hookrightarrow J(S^2)$ IS ISO ON H^i FOR ALL $i \leq 2m$

HENCE: FIXING m, WE CAN PROVE THE STATEMENT FOR J_m INSTEAD OF J.

CONSEQUENCE OF KÜNNETH:

$$H^*((S^2)^m; \mathbb{Z}) \cong \mathbb{Z}[\alpha_1, \dots, \alpha_m] \big/ (\alpha_1^2, \dots, \alpha_m^2)$$

WITH $\alpha_i := p_i^*(\text{GEN OF } H^2(S^2; \mathbb{Z}))$

PROJECTION ON i-TH FACTOR

CLAIM: 1) $q^*(x_2) = \alpha_1 + \dots + \alpha_m$ (FOR SUITABLE CHOICE OF GEN)

2) $q_*(x_{2m}) = \alpha_1 \cdot \dots \cdot \alpha_m$

SKETCH: 1) LET $j_i: S^2 \longrightarrow (S^2)^m$ BE INCLUSION AS i-TH FACTOR (OTHER COORDINATES $= p$)

THEN $q \circ j_i$ MAPS THE 2-CELL OF S^2 TO THE 2-CELL OF $J_m(S^2)$ WITH DEGREE 1 (MORE PRECISELY: AFTER COLLAPSIN ∂(2-CELL))

$\Rightarrow$ BY CELLULAR COHOMOLOGY, $(q \circ j_i)^*(x_2)$ GEN OF $H^2(S^2; \mathbb{Z})$

KÜNNETH $\Rightarrow$ THIS IS ENOUGH TO CONCLUDE

2) $\alpha_1 \cdot \dots \cdot \alpha_m$ IS GEN OF $H^{2m}((S^2)^m; \mathbb{Z})$, AND q^* IS ISO ON H^{2m} $\square$

HENCE WE HAVE

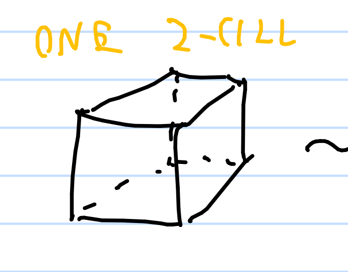
$$q^*(x_2^m) = (\alpha_1 + \dots + \alpha_m)^m \underset{\uparrow}{=} m! \, \alpha_1 \cdots \alpha_m = m! \, q^*(x_{2m})$$

SINCE $\alpha_i^2 = 0$, ALL OTHER TERMS ARE 0

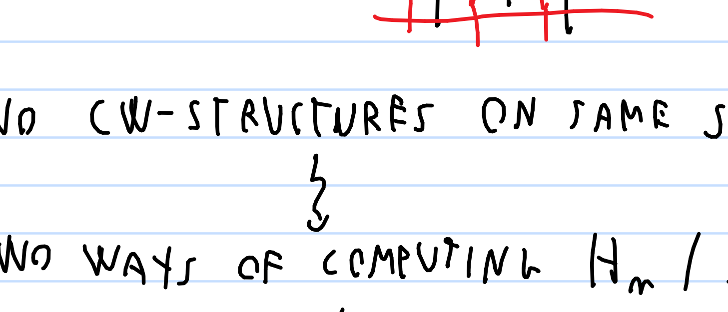
BUT q^* IS ISO ON H^{2m} $\Rightarrow$ $x_2^m = m! \, x_{2m}$ $\square$

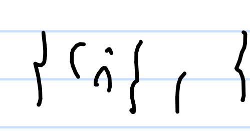
AFTER EASTER: HOMOLOGY AND COHOMOLOGY OF MANIFOLDS

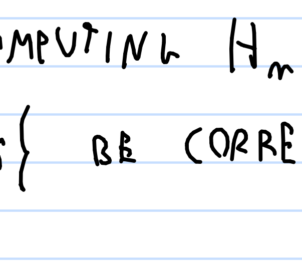
SOME PICTURES: DUAL CELL CPLXS



ONE 0-CELL FOR EACH 2-CELL
ONE 1-CELL FOR EACH 1-CELL
ONE 2-CELL FOR EACH 2-CELL







TWO CW-STRUCTURES ON SAME SPACE
$\rightsquigarrow$
TWO WAYS OF COMPUTING H_m / H^m

LET $\{C_i\}$, $\{\check{C}_i\}$ BE CORRESPONDING CHAIN CPLXS.

$\check{C}_{2-i}$ IS "ALMOST" CANONICALLY ISOMORPHIC TO C_i^* $\searrow$ SIGN ISSUE, IGNORE THAT..

THE ISO IS COMPATIBLE WITH ∂ AND δ
$\rightsquigarrow$
C_i^* COMPUTES BOTH H^i AND H_{2-i}!

SOMETIMES THIS ACTUALLY WORKS, AND WE'LL PROVE:

THM (POINCARÉ DUALITY) LET M BE CPT ORIENTABLE MFLD. THEN

$$H^i(M; \mathbb{Z}) \cong H_{m-i}(M; \mathbb{Z})$$