NEW CHAPTER: ALGEBRAIC TOPOLOGY OF MANIFOLDS

RECALL: AN m-MANIFOLD, OR MFLD OF DIM m, IS A HAUSDORFF, <u>2ND COUNTABLE</u> TOP SPACE S.T. EVERY POINT HAS A NEIGH HOMEO TO $\mathbb{R}^m$

NOT IN HATCHER, BUT USUALLY ASSUMED

E.G.: S^m, $\mathbb{RP}^m$, $\mathbb{CP}^{m/2}$, $(S^1)^m$, ...

NOTE: THE DIMENSION CAN BE DETECTED BY HOMOLOGY SINCE:

$$H_i(M, M-\{x\}; \mathbb{Z}) \overset{\text{EXCISION}}{\cong} H_i(\mathbb{R}^m, \mathbb{R}^m - \{0\}; \mathbb{Z})$$

$$\overset{\mathbb{R}^m \text{ CONTRACTIBLE} + \text{LES OF PAIR}}{\cong} \tilde{H}_{i-1}(\mathbb{R}^m - \{0\}; \mathbb{Z})$$

$$\cong \tilde{H}_{i-1}(S^{m-1}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = m \\ 0 & i \neq m \end{cases}$$

$\mathbb{R}^m - \{0\} \simeq S^{m-1}$

CONVENTION: IF WE DON'T WRITE HOM. COEFF, THEN IT'S $\mathbb{Z}$.

KEY NOTION: ORIENTABILITY

LET'S START WITH $\mathbb{R}^m$. IDEA:

ORIENTATION ON $\mathbb{R}^m$ AT x = "THING" THAT IS PRESERVED BY ROTATIONS, AND REVERSED BY REFLECTIONS + WHICH ONLY DEPENDS ON A NEIGH OF x

THERE ARE SEVERAL POSSIBLE "THINGS" THAT DO THIS, ALL RELATED TO EACH OTHER.

FOR US, IT'S BEST TO USE A HOMOLOGICAL "THING":

DEFN: AN ORIENTATION OF $\mathbb{R}^m$ AT x IS A CHOICE OF GENERATOR OF $H_m(\mathbb{R}^m, \mathbb{R}^m - \{x\})$

LEMMA: α ORIENT AT x. THEN $f_*(\alpha) = \alpha$ IF f IS ROTATION AROUND x, AND $f_*(\alpha) = -\alpha$ IF f IS REFLECTION FIXING x

PROOF: SAY $x = 0$. THE ISO IN COMPUTATION ABOVE ARE NATURAL, SO WE HAVE:

$$\begin{tikzcd} H_m(\mathbb{R}^m, \mathbb{R}^m - \{0\}) \arrow[r] \arrow[d, "f_*"] \arrow[dr, phantom, "\circlearrowright"] & \tilde{H}_{m-1}(S^{m-1}) \arrow[d, "(f|_{S^{m-1}})_*"] \\ H_m(\mathbb{R}^m, \mathbb{R}^m - \{0\}) \arrow[r] & \tilde{H}_{m-1}(S^{m-1}) \end{tikzcd}$$

HENCE, WE CAN USE RESULTS WE KNOW ABOUT DEGREES

IMPORTANT OBSERVATION:

AN ORIENT. AT x DETERMINES AN ORIENT AT y USING THE FOLLOWING ISO INDUCED BY INCLUSIONS:

$$H_m(\mathbb{R}^m, \mathbb{R}^m - \{x\}) \xleftarrow{\cong} H_m(\mathbb{R}^m, \mathbb{R}^m - B) \xrightarrow{\cong} H_m(\mathbb{R}^m, \mathbb{R}^m - \{y\})$$

B = BALL CONTAINING BOTH x AND y

DEFN: M m-MANIFOLD, A <u>LOCAL ORIENTATION</u> AT x IS A GENERATOR OF $H_m(M, M - \{x\})$

NOTATION: $\cdot\ H_m(X, X - A; G) = H_m(X|A; G)$

$\cdot\ H_m(X|x) = H_m(X|\{x\})$

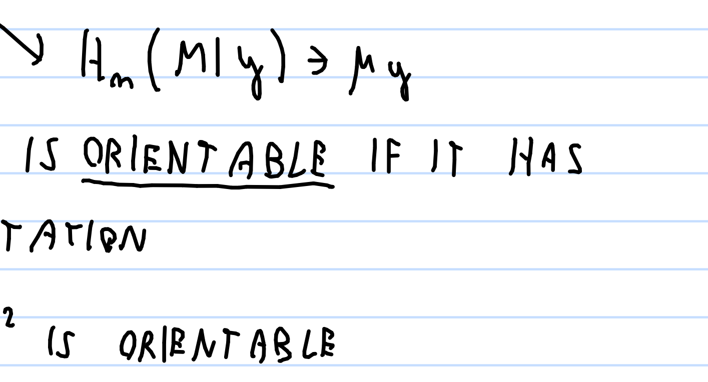
$H_m(X|A; G)$ IS CALLED <u>LOCAL HOMOLOGY</u> AT A

NOTE: BY EXCISION IT ONLY DEPENDS ON NEIGH'S OF A

IDEA: AN ORIENTATION IS A COHERENT CHOICE OF LOCAL ORIENTATIONS AT ALL POINTS

DEFN: AN <u>ORIENTATION</u> OF THE m-MFLD M IS A FUNCTION $x \mapsto \mu_x$ ASSIGNING TO EACH x A LOCAL ORIENT. AT x S.T. EVERY x HAS A NEIGH HOMEO TO $\mathbb{R}^m$ CONTAINING AN OPEN BALL B AROUND x S.T. $\exists$ GEN μ_B OF $H_m(M|B)$ S.T. $\forall y \in B$ WE HAVE $\mu_y = i_*(\mu_B)$, WHERE $i: (M, M-B) \to (M, M - \{y\})$ INCLUSION.

PICTURE:



$$\begin{tikzcd} H_m(M|B) \arrow[r] \arrow[dr] & H_m(M|x) \ni \mu_x \\ & H_m(M|y) \ni \mu_y \end{tikzcd}$$

DEFN: A MFLD IS <u>ORIENTABLE</u> IF IT HAS AN ORIENTATION

WE'LL SEE: $\cdot\ S^2$ IS ORIENTABLE

$\cdot\ \mathbb{RP}^2$ IS NOT ORIENTABLE

WE HAVE A 2-SHEETED COVER $S^2 \to \mathbb{RP}^2$.

$|p^{-1}(\{x\})| = 2 \quad \forall x$

THIS IS AN EXAMPLE OF A GENERAL FACT:

PROP: 1) EVERY MFLD M HAS AN ORIENTABLE 2-SHEETED COVER

2) M CONN. THEN M IS ORIENTABLE $\Leftrightarrow$ $\tilde{M}$ HAS 2 CONN. COMP

BEFORE PROVING PROP, NOTE THAT

COR 1) M SIMPLY CONNECTED $\Rightarrow$ M ORIENTABLE

2) MORE GENERALLY, M CONN + $\pi_1(M)$ HAS NO INDEX-2 SUBGPS $\Rightarrow$ M IS ORIENTABLE

PROOF OF COR USES: $p: X \to Y$ 2-SHEETED COVER, X PATH-CONN $\Rightarrow$ $p_*(\pi_1(X)) < \pi_1(Y)$ HAS INDEX 2.

OUTLINE OF PROOF OF PROP:

AS A SET:

GEN OF $H_m(M|x) \cong \mathbb{Z}$

$\tilde{M} = \{\mu_x \mid x \in M, \mu_x \text{ LOCAL ORIENT AT } x\}$

$p: \tilde{M} \to M$, $\mu_x \mapsto x$ IS A 2-TO-1 SURJECTION

LEFT TO DO: DEFINE A TOPOLOGY ON $\tilde{M}$ S.T. p IS COVERING MAP

WE DEFINE A BASIS $\{U(\mu_B)\}$ AS FOLLOWS:

GIVEN AN OPEN BALL $B \subseteq \mathbb{R}^m \subseteq M$, AND A GEN $\mu_B \in H_m(M|B)$, LET $U(\mu_B)$ BE THE SET OF ALL $i_*(\mu_B)$ FOR $i: (M, M-B) \to (M, M - \{x\})$ INCLUSION (NOTE: $x \in B$)

THINGS THAT SHOULD BE CHECKED:

1) $\{U(\mu_B)\}$ IS BASIS

2) p IS COVERING MAP ($\Rightarrow \tilde{M}$ IS MFLD)

3) $\tilde{M}$ IS ORIENTABLE

REGARDING 3: $U(\mu_B) \to B$, $\mu_y \mapsto y$ IS HOMEO $\Rightarrow$

$$H_m(\tilde{M}|\mu_x) \cong H_m(U(\mu_B)|\mu_x) \cong H_m(B|x) \cong H_m(M|x) \ni \mu_x$$

"CANONICAL" LOCAL ORIENTATION OF $\tilde{M}$ AT μ_x = IMAGE OF μ_x THROUGH THE ISO $\longrightarrow$

THE "B" IN THE DEFN OF ORIENT CAN BE TAKEN TO BE A $U(\mu_B)$ $\square$