

# R-ORIENTABILITY

THERE'S A LARGER (MORE "CANONICAL") COVERING SPACE THAN  $\tilde{M}$

$$M_{\mathbb{Z}} \cong \left\{ \alpha_x \in H_m(M|x) \right\}$$

AS A SET

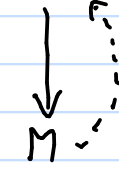
TOPOLOGY = SIMILAR DEFN TO  $\tilde{M}$

NOTE:  $M_0 = \{ \alpha \in H_m(M|x) \}$  IS A COPY OF  $M$

$M_{\mathbb{Z}}$  IS A DISJOINT UNION OF  $M_0$  AND COPIES  $M_k$  OF  $\tilde{M}$ ,  $k=1, \dots$

$$M_k = \{ k \cdot (\text{GEN OF } H_m(M|x)) \}$$

DEFN: A CONT MAP  $M \rightarrow M_{\mathbb{Z}}$  OF THE FORM  $x \mapsto \alpha_x$  IS CALLED A SECTION



NOTE: ORIENTATION  $\Leftrightarrow$  SECTION  $x \mapsto \alpha_x$  S.T. EVERY  $\alpha_x$  IS GEN.

$(M \text{ ORIENTABLE} \Leftrightarrow \exists \text{ SECTION OTHER THAN } 0\text{-SECTION})$

$R =$  COMMUTATIVE RING WITH 1

ONE CAN DEFINE  $R$ -ORIENTATIONS, INTERPRETING "GENERATOR" AS "UNIT"

$$x \mapsto \alpha_x \in H_m(M|x; R) \cong R, \alpha_x \text{ UNIT.}$$

$$M_R = \text{SIMILAR DEFN} = \{ \alpha_x \in H_m(M|x; R) \}$$

FACT:  $H_m(M|x; R) \cong H_m(M|x) \otimes R$ , CANONICALLY

$\rightsquigarrow$  FOR  $\alpha \in R$ , WE HAVE SUBCOVER

$$M_\alpha \rightarrow M, \text{ WHERE}$$

$$M_\alpha = \{ \pm \mu_x \otimes \alpha, \mu_x \text{ GEN OF } H_m(M|x) \}$$

$M_\alpha \cong M$  IF  $\alpha = -\alpha$  (CPR WITH  $\mu_0$ )

$M_\alpha \cong \tilde{M}$  IF  $\alpha \neq -\alpha$

AS A COVER:  $\begin{matrix} \hat{X}_1 & \rightarrow & \hat{X}_2 \\ & \searrow & \swarrow \\ & X & \end{matrix}$

IN PART:

$M$  ORIENTABLE  $\Rightarrow M$   $R$ -ORIENTABLE

INDEED, FOR ANY UNIT  $\alpha$ ,  $\exists$  SECTION  $M \rightarrow M_\alpha$ , SINCE  $M_\alpha \cong \tilde{M}$

IF  $R$  HAS UNIT OF ORDER 2, THEN ANY  $M$  IS  $R$ -ORIENTABLE

IN PART: EVERY MFLD IS  $\mathbb{Z}/2$ -ORIENTABLE

BASICALLY: THE INTERESTING CASES ARE  $R = \mathbb{Z}$  AND  $R = \mathbb{Z}/2$

SOMETIMES IN LITERATURE "MANIFOLD" ACTUALLY MEANS "MANIFOLD WITH BOUNDARY".

JUST TO MAKE SURE:

DEFN:  $M$  MANIFOLD IS CLOSED IF IT IS COMPACT AND HAS NO BOUNDARY

THM: LET  $M$  BE CLOSED CONNECTED  $m$ -MANIFOLD. THEN:

a)  $M$   $R$ -ORIENTABLE  $\Rightarrow$  THE MAP  $H_m(M; R) \rightarrow H_m(M|x; R) \cong R$  IS  $\neq 0 \forall x$

b)  $M$  NOT  $R$ -ORIENT  $\Rightarrow H_m(M; R) \rightarrow H_m(M|x; R)$  IS INS WITH IMAGE  $\{ \alpha \mid 2\alpha = 0 \} \forall x$

c)  $H_i(M; R) = 0 \forall i > m$

IN PART:  $H_m(M; \mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ ORIENT.} \\ 0 & M \text{ NOT ORIENT.} \end{cases}$

$$H_m(M; \mathbb{Z}/2) = \mathbb{Z}/2$$

COR:  $T^2$  IS ORIENTABLE,  $\mathbb{R}P^2$  AND KLEIN BOTTLE ARE NOT.

PROOF: NEXT WEEK

CONSEQUENCE OF THM:

DEFN:  $M$   $m$ -MFLD. A FUNDAMENTAL CLASS FOR  $M$  WITH COEFF IN  $R$  IS AN ELEMENT OF  $H_m(M; R)$  WHOSE IMAGE IN ALL  $H_m(M|x; R)$  IS A GENERATOR.

PROP:  $M$  HAS A FUND. CLASS  $\Leftrightarrow M$  IS CLOSED AND  $R$ -ORIENT

PROOF:  $\Leftarrow$  FOLLOWS FROM a) (APPLIED TO CONN. COMP)

$\Rightarrow$ :  $\mu =$  FUND CLASS,  $\mu_x =$  IMAGE IN  $H_m(M|x; R)$

THEN  $x \mapsto \mu_x$  IS  $R$ -ORIENT.

(EXERCISE: IT'S CONTINUOUS)

ALSO, IF  $\mu = \left[ \sum c_i \sigma_i \right]$  THEN  $U \text{im}(\sigma_i) = M$

$\Rightarrow M$  IS CPT  $\square$