R-ORIENTABILITY

THERE'S A LARGER (MORE "CANONICAL") COVERING SPACE THAN $\tilde{M}$

$M_{\mathbb{Z}} \underset{\substack{\uparrow \\ \text{AS A SET}}}{=} \{\alpha_x \in H_m(M|x)\}$

TOPOLOGY = SIMILAR DEFN TO $\tilde{M}$

NOTE: · $M_0 = \{0 \in H_m(M|x)\}$ IS A COPY OF M

· $M_{\mathbb{Z}}$ IS A DISJOINT UNION OF M_0 AND COPIES M_k OF $\tilde{M}$, $k = 1, \dots$

$$M_k = \{k \cdot (\text{GEN OF } H_m(M|x)\}$$

DEFN: A CONT MAP $M \to M_{\mathbb{Z}}$ OF THE FORM $x \mapsto \alpha_x$ IS CALLED A <u>SECTION</u>

$$\begin{tikzcd} M_{\mathbb{Z}} \arrow[d] \\ M \arrow[u, dashed, bend right] \end{tikzcd}$$

NOTE: · ORIENTATION $\Leftrightarrow$ SECTION $x \mapsto \alpha_x$ S.T. EVERY α_x IS GEN.

(M ORIENTABLE $\Leftrightarrow$ $\exists$ SECTION OTHER THAN 0-SECTION)

R = COMMUTATIVE RING WITH 1

ONE CAN DEFINE R-ORIENTATIONS, INTERPRETING "GENERATOR" AS "UNIT"

$x \mapsto \alpha_x \in H_m(M|x;R) \cong R$, α_x UNIT.

M_R = SIMILAR DEFN = $\{\alpha_x \in H_m(M|x;R)\}$

FACT: $H_m(M|x;R) \cong H_m(M|x) \otimes R$, CANONICALLY

$\rightsquigarrow$ FOR $r \in R$, WE HAVE SUBCOVER $M_r \to M$, WHERE

$M_r = \{\pm\mu_x \otimes r,\ \mu_x \text{ GEN OF } H_m(M|x)\}$

$$M_r \underset{\substack{\uparrow \\ \text{AS A COVER}}}{\cong} \begin{cases} M & \text{IF } r = -r \\ \tilde{M} & \text{IF } r \neq -r \end{cases}$$

(CFR WITH M_0

AS A COVER: $\begin{tikzcd} \hat{X}_1 \arrow[r] \arrow[dr] & \hat{X}_2 \arrow[d] \\ & X \end{tikzcd}$

IN PART:

· M ORIENTABLE $\Rightarrow$ M R-ORIENTABLE
INDEED, FOR ANY UNIT r, $\exists$ SECTION $M \to M_r$, SINCE $M_r \cong \tilde{M}$

· IF R HAS UNIT OF ORDER 2, THEN <u>ANY</u> M IS R-ORIENTABLE

IN PART: $\boxed{\text{EVERY MFLD IS } \mathbb{Z}/2\text{-ORIENTABLE}}$

BASICALLY: THE INTERESTING CASES ARE $R = \mathbb{Z}$ AND $R = \mathbb{Z}/2$

SOMETIMES IN LITERATURE "MANIFOLD" ACTUALLY MEANS "MANIFOLD WITH BOUNDARY".

JUST TO MAKE SURE:

DEFN: M MANIFOLD IS <u>CLOSED</u> IF IT IS COMPACT AND HAS NO BOUNDARY

THM: LET M BE CLOSED CONNECTED m-MANIFOLD. THEN:

a) M R-ORIENTABLE $\Rightarrow$ THE MAP $H_m(M;R) \to H_m(M|x;R) \cong R$ IS ISO $\forall x$

b) M NOT R-ORIENT $\Rightarrow$ $H_m(M;R) \to H_m(M|x;R)$ IS INJ WITH IMAGE $\{r \mid 2r = 0\}$ $\forall x$

c) $H_i(M;R) = 0 \quad \forall i > m$

IN PART: · $H_m(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ ORIENT.} \\ 0 & M \text{ NOT ORIENT.} \end{cases}$

$$\cdot\ H_m(M;\mathbb{Z}/2) = \mathbb{Z}/2$$

COR: T^2 IS ORIENTABLE, $\mathbb{RP}^2$ AND KLEIN BOTTLE ARE NOT.

PROOF: NEXT WEEK

CONSEQUENCE OF THM:

DEFN: M m-MFLD. A <u>FUNDAMENTAL CLASS</u> FOR M WITH COEFF IN R IS AN ELEMENT OF $H_m(M;R)$ WHOSE IMAGE IN ALL $H_m(M|x;R)$ IS A GENERATOR.

PROP: M HAS A FUND. CLASS $\Leftrightarrow$ M IS CLOSED AND R-ORIENT

PROOF: $\Leftarrow$ FOLLOWS FROM a) (APPLIED TO CONN. COMP)

$\Rightarrow$: μ = FUND CLASS, μ_x = IMAGE IN $H_m(M|x;R)$
THEN $x \mapsto \mu_x$ IS R-ORIENT.
(EXERCISE: IT'S CONTINUOUS)

ALSO, IF $\mu = [\sum c_i \sigma_i]$ THEN $\bigcup \mathrm{im}(\sigma_i) = M$
$\Rightarrow$ M IS CPT $\square$