

GOAL = TO PROVE:

THM: LET M BE CLOSED CONNECTED m -MANIFOLD. THEN:

- a) M \mathbb{R} -ORIENTABLE \Rightarrow THE MAP $H_m(M; \mathbb{R}) \rightarrow H_m(M/x; \mathbb{R}) \cong \mathbb{R}$ IS ISO $\forall x$
- b) M NOT \mathbb{R} -ORIENT $\Rightarrow H_m(M; \mathbb{R}) \rightarrow H_m(M/x; \mathbb{R})$ IS INS WITH IMAGE $\{z \mid zz=0\} \forall x$
- c) $H_i(M; \mathbb{R}) = 0 \forall i > m$

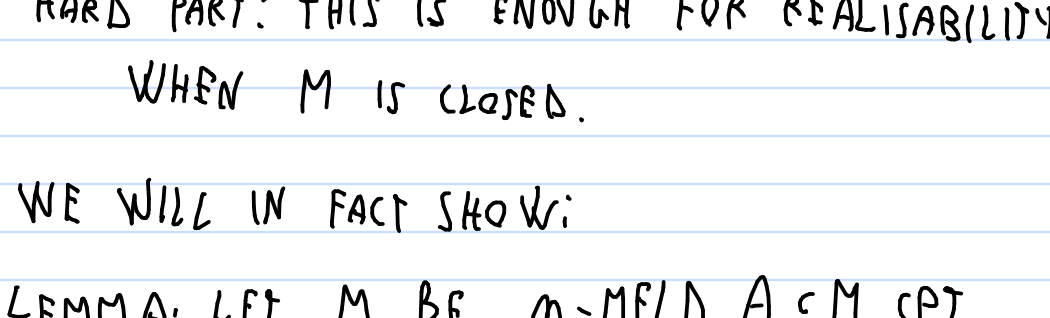
POINT OF THE THM

AN ORIENTATION GIVES $x \mapsto \alpha_x \in H_m(M/x; \mathbb{R})$.

THE GOAL IS TO START FROM SUCH SECTION AND CONSTRUCT $\alpha \in H_m(M; \mathbb{R})$ REALISING IT,

MEANING: $H_m(M; \mathbb{R}) \rightarrow H_m(M/x; \mathbb{R}) \forall x$
 $\alpha \mapsto \alpha_x$

NOTE: IF $x \mapsto \alpha_x$ IS REALISABLE, THEN IT SATISFIES THE "CONSISTENCY COND":



IN FACT: $H_m(M; \mathbb{R}) \rightarrow H_m(M/x; \mathbb{R})$
 \searrow
 $H_m(M/B; \mathbb{R}) \nearrow$

HARD PART: THIS IS ENOUGH FOR REALISABILITY, WHEN M IS CLOSED.

WE WILL IN FACT SHOW:

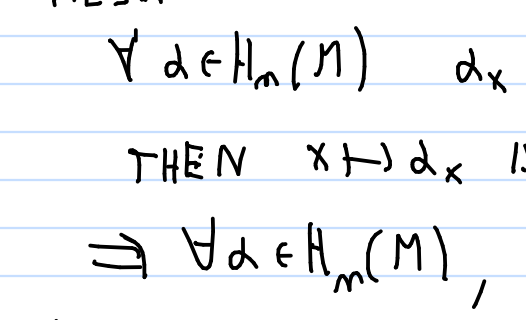
LEMMA: LET M BE m -MFLD, $A \subseteq M$ CPT

THEN: 1) LET $x \mapsto \alpha_x$ BE SECTION OF COVER $M_R \rightarrow M$. THEN $\exists!$ CLASS $\alpha_A \in H_m(M/A; \mathbb{R})$ WHICH MAPS TO α_x

$\forall x \in A$

2) $H_i(M/A; \mathbb{R}) = 0 \forall i > m$

RECALL $M_R = \{ \alpha_x \in H_m(M/x; \mathbb{R}) \}$

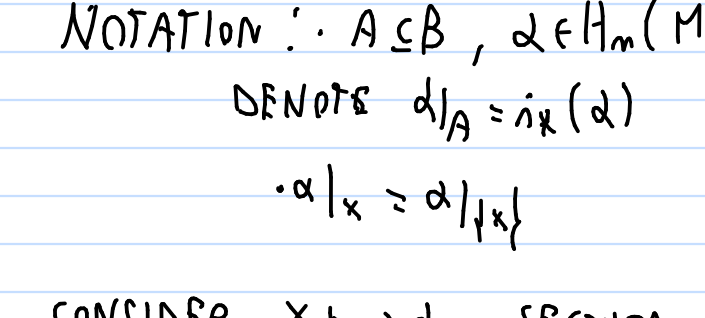


PROOF OF (LEMMA \Rightarrow THM), FOR $\mathbb{R} = \mathbb{Z}$:

RMK (FROM COVERING THEORY): IF M CONN, \exists A SECTION MAPPING x TO α_x , THEN THE SECTION IS UNIQUE.

a) M ORIENTABLE $\Rightarrow \forall \alpha_x \exists$ SECTION WITH $x \mapsto \alpha_x$, SINCE $M_{\mathbb{Z}} = \coprod M$

SO: $H_m(M) \leftrightarrow$ SECTIONS $\leftrightarrow H_m(M/x)$



b) M NOT ORIENTABLE, M CONNECTED \Rightarrow ONLY SECTION IS 0-SECTION

ALSO. $\forall \alpha \in H_m(M) \alpha_x$ IMAGE IN $H_m(M/x)$

THEN $x \mapsto \alpha_x$ IS SECTION.

$\Rightarrow \forall \alpha \in H_m(M), \alpha = 0 \checkmark$

c) THIS IS PART 2 OF LEMMA & PROOF OF LEMMA:

\mathbb{R} PLAYS NO ROLE, SO WE OMIT IT.

STEP 1: IF LEMMA HOLDS FOR $A, B, A \cap B$, THEN IT HOLDS FOR $A \cup B$

CONSIDER MAYER VECTORS, WITH $i \geq m$:

$$0 \rightarrow H_i(M/A \cup B) \rightarrow H_i(M/A) \oplus H_i(M/B) \rightarrow H_i(M/A \cap B) \rightarrow H_{i-1}(M/A \cup B)$$

FOR $i > m$: $(*) = 0 \Rightarrow H_i(M/A \cup B) = 0$, THAT IS, PART 2

PART 1:

NOTATION: $\cdot A \subseteq B, \alpha \in H_m(M/B)$, DENOTE $\alpha|_A = \alpha_A$

$$\alpha|_x = \alpha|_{\{x\}}$$

CONSIDER $x \mapsto \alpha_x$ SECTION, AND $\alpha_A, \alpha_B, \alpha_{A \cap B}$ S.T. $\alpha_A|_x = \alpha_x$, ETC.

$$0 \rightarrow H_m(M/A \cup B) \rightarrow H_m(M/A) \oplus H_m(M/B) \rightarrow H_m(M/A \cap B)$$

BY UNIQUENESS OF $\alpha_{A \cap B}$, WE HAVE

$$\alpha_A|_{A \cap B} = \alpha_{A \cap B}$$

$$\alpha_B|_{A \cap B} = \alpha_{A \cap B}$$

$(*) \Rightarrow \exists \alpha_{A \cup B}$ WITH $\cdot \alpha_{A \cup B}|_A = \alpha_A$

$\cdot \alpha_{A \cup B}|_B = \alpha_B$

BY INJECTIVITY, IT'S UNIQUE

EASY CHECK: $\alpha_{A \cup B}$ IS WHAT WE'RE LOOKING FOR

STEP 2: IF LEMMA HOLDS FOR \mathbb{R}^m , THEN IT HOLDS FOR ANY M

IN FACT, ANY CPT A IS FINITE UNION $A_1 \cup \dots \cup A_m$, EACH CONTAINED IN A COPY OF \mathbb{R}^m

$m=1$: USE EXCISION

$m>1$: DO INDUCTION, USING STEP 1.

STEP 3: $M = \mathbb{R}^m, A = A_1 \cup \dots \cup A_m, A_i$ IS CONVEX CPT SET

$m=1$: $H_i(\mathbb{R}^m/A) \rightarrow H_i(\mathbb{R}^m/x)$ IS ISO BY HOMOTOPY EQUIVALENCE

$m>1$: INDUCTION AS IN STEP 2

STEP 4: $M = \mathbb{R}^m$, ARBITRARY A

PART 1 "3": USE α_B FOR ANY BALL $B \supseteq A$

OTHER PARTS (SEE PIC BELOW):

LET $\alpha = [\zeta] \in H_i(\mathbb{R}^m/A)$, THEN $\delta \zeta = \sum c_j \sigma_j$ AND $\cup \text{im}(\sigma_j) \cap A = \emptyset$

$\Rightarrow A \subseteq B_1 \cup \dots \cup B_m = B, B_k$ CLOSED BALLS $\cup \text{im}(\sigma_j) \cap B = \emptyset$ (FOR LATER: $B_i \cap A \neq \emptyset$)

$\Rightarrow \zeta$ REPRESENTS A CLASS $\alpha_B \in H_i(\mathbb{R}^m/B)$

AND $\alpha_B|_A = [\zeta] = \alpha$

$i > m$: BY STEP 3, $\alpha_B = 0 \Rightarrow \alpha = 0$

$i = m$: IF $\alpha_1, \alpha_2 \in H_m(\mathbb{R}^m/A)$ ARE S.T. $\alpha_i|_x = \alpha_x \forall x$, THEN DO THE ABOVE WITH $\alpha = \alpha_1 - \alpha_2$. CAN CONCLUDE AS ABOVE, BUT:

SUBTLETY: NEED $\alpha_B|_x = 0 \forall x \in \underline{B}$

FOR $x \in A$, CLEAR: $\alpha|_x = (\alpha_1 - \alpha_2)|_x$

OTHERWISE: $x \in B_i, B_i \cap A \ni y$

OK SINCE: $(\alpha|_{B_i})|_y = 0$, AND ALSO

$|_y: H_m(\mathbb{R}^m/B_i) \rightarrow H_m(\mathbb{R}^m/y)$ IS ISO. D

