GOAL = TO PROVE:

THM: LET M BE CLOSED CONNECTED m-MANIFOLD. THEN:

a) M R-ORIENTABLE $\Rightarrow$ THE MAP $H_m(M;R) \to H_m(M|x;R) \cong R$ IS ISO $\forall x$

b) M NOT R-ORIENT $\Rightarrow$ $H_m(M;R) \to H_m(M|x;R)$ IS INJ WITH IMAGE $\{r \mid 2r=0\}$ $\forall x$

c) $H_i(M;R) = 0$ $\forall i > m$

POINT OF THE THM

AN ORIENTATION GIVES $x \mapsto \alpha_x \in H_m(M|x;R)$.

THE GOAL IS TO START FROM SUCH SECTION AND CONSTRUCT $\alpha \in H_m(M;R)$ REALISING IT, MEANING:

$$\begin{array}{ccc} H_m(M;R) & \longrightarrow & H_m(M|x;R) \quad \forall x \\ \alpha & \longmapsto & \alpha_x \end{array}$$

NOTE: IF $x \mapsto \alpha_x$ IS REALISABLE, THEN IT SATISFIES THE "CONSISTENCY COND":

$B \subseteq \mathbb{R}^n \subseteq M$, $x, y \in B$ $\Rightarrow$
$$\begin{tikzcd} & H_m(M|x;R) \ni \alpha_x \\ \alpha_B \in H_m(M|B;R) \arrow[ur] \arrow[dr] & \\ & H_m(M|y;R) \ni \alpha_y \end{tikzcd}$$

IN FACT:
$$\begin{tikzcd} H_m(M;R) \arrow[rr] \arrow[dr] & & H_m(M|x;R) \\ & H_m(M|B;R) \arrow[ur] & \end{tikzcd}$$

HARD PART: THIS IS ENOUGH FOR REALISABILITY, WHEN M IS CLOSED.

WE WILL IN FACT SHOW:

LEMMA: LET M BE m-MFLD, $A \subseteq M$ CPT

THEN: 1) LET $x \mapsto \alpha_x$ BE SECTION OF COVER $M_R \to M$. THEN $\exists!$ CLASS $\alpha_A \in H_m(M|A;R)$ WHICH MAPS TO α_x $\forall x \in A$

2) $H_i(M|A;R) = 0$ $\forall i > m$

RECALL $M_R = \{\alpha_x \in H_m(M|x;R)\}$

SECTION:
$$\begin{tikzcd} M_R \arrow[d] \\ M \arrow[u, bend right, dashed] \end{tikzcd}$$

PROOF OF (LEMMA $\Rightarrow$ THM), FOR $R = \mathbb{Z}$:

RMK (FROM COVERING THEORY): IF M CONN, $\exists$ A SECTION MAPPING x TO α_x, THEN THE SECTION IS UNIQUE.

a) M ORIENTABLE $\Rightarrow$ $\forall \alpha_x$ $\exists$ SECTION WITH $x \to \alpha_x$, SINCE $M_{\mathbb{Z}} = \sqcup M$

SO: $\underset{\substack{\uparrow \\ \text{LEMMA,} \\ A = M}}{H_m(M) \leftrightarrow} \text{SECTIONS} \underset{\substack{\uparrow \\ \text{SEE} \\ \text{ABOVE}}}{\leftrightarrow} H_m(M|x)$

b) M NOT ORIENTABLE, M CONNECTED $\Rightarrow$ ONLY SECTION IS 0-SECTION

ALSO.

$\forall \alpha \in H_m(M)$ α_x IMAGE IN $H_m(M|x)$

THEN $x \mapsto \alpha_x$ IS SECTION.

$\Rightarrow \forall \alpha \in H_m(M)$, $\alpha = 0$ ✓

c) THIS IS PART 2 OF LEMMA □

PROOF OF LEMMA:

R PLAYS NO ROLE, SO WE OMIT IT.

STEP 1: IF LEMMA HOLDS FOR $A, B, A \cap B$, THEN IT HOLDS FOR $A \cup B$

CONSIDER MAYER VIETORIS, WITH $i \geq m$:

$$\underset{\substack{\uparrow \\ = H_{i+1}(M|A\cap B)}}{0} \to H_i(M|A\cup B) \to \underset{(*)}{H_i(M|A) \oplus H_i(M|B)} \to H_i(M|A\cap B)$$

FOR $i > m$: $(*) = 0 \Rightarrow H_i(M|A \cup B) = 0$, THAT IS, PART 2

PART 1:

NOTATION: $\cdot$ $A \subseteq B$, $\alpha \in H_m(M|B)$, DENOTE $\alpha|_A = i_*(\alpha)$

$\cdot$ $\alpha|_x = \alpha|_{\{x\}}$

CONSIDER $x \mapsto \alpha_x$ SECTION, AND $\alpha_A, \alpha_B, \alpha_{A\cap B}$ S.T. $\alpha_A|_x = \alpha_x$, ETC.

$$0 \to H_m(M|A\cup B) \to \overset{(*)}{\underset{\substack{\Downarrow \\ \alpha_A \quad\quad \alpha_B}}{H_m(M|A) \oplus H_m(M|B)}} \to \underset{\substack{\Downarrow \\ \alpha_{A\cap B}}}{H_m(M|A\cap B)}$$

BY UNIQUENES OF $\alpha_{A\cap B}$, WE HAVE

$\alpha_A|_{A\cap B} = \alpha_{A\cap B}$

$\alpha_B|_{A\cap B} = \alpha_{A\cap B}$

$(*) \Rightarrow \exists\, \alpha_{A\cup B}$ WITH $\cdot$ $\alpha_{A\cup B}|_A = \alpha_A$

$\cdot$ $\alpha_{A\cup B}|_B = \alpha_B$

BY INJECTIVITIVITY, IT'S UNIQUE

EASY CHECK: $\alpha_{A\cup B}$ IS WHAT WE'RE LOOKING FOR

STEP 2: IF LEMMA HOLDS FOR $\mathbb{R}^m$, THEN IT HOLDS FOR ANY M

IN FACT, ANY CPT A IS FINITE UNION $A_1 \cup \ldots \cup A_m$, EACH CONTAINED IN A COPY OF $\mathbb{R}^n$

$m = 1$: USE EXCISION

$m > 1$: DO INDUCTION, USING STEP 1.

STEP 3: $M = \mathbb{R}^n$, $A = A_1 \cup \ldots \cup A_m$, A_i IS CONVEX CPT SET

$m = 1$: $H_i(\mathbb{R}^n|A) \to H_i(\mathbb{R}^n|x)$ IS ISO BY HOMOTOPY EQUIVALENCE

$m > 1$: INDUCTION AS IN STEP 2

STEP 4: $M = \mathbb{R}^n$, ARBITRARY A

PART 1, "3": USE α_B FOR ANY BALL $B \supseteq A$

OTHER PARTS (SEE PIC BELOW):

LET $\alpha = [z] \in H_i(\mathbb{R}^n|A)$, THEN

$\partial z = \sum c_j \sigma_j$ AND $\bigcup im(\sigma_j) \cap A = \emptyset$

$\Rightarrow A \subseteq B_1 \cup \ldots \cup B_m = B$, B_k CLOSED BALLS

$\bigcup im(\sigma_j) \cap B = \emptyset$ FOR LATER: $B_k \cap A \neq \emptyset$

$\Rightarrow z$ REPRESENTS A CLASS $\alpha_B \in H_j(\mathbb{R}^n|B)$

AND $\alpha_B|_A = [z] = \alpha$

$i > m$: BY STEP 3, $\alpha_B = 0 \Rightarrow \alpha = 0$

$i = m$: IF $\alpha_1, \alpha_2 \in H_m(\mathbb{R}^m|A)$ ARE S.T. $\alpha_i|_x = \alpha_x$ $\forall x$, THEN DO THE ABOVE WITH $\alpha = \alpha_1 - \alpha_2$. CAN CONCLUDE AS ABOVE, BUT:

SUBTLETY: NEED $\alpha_B|_x = 0$ $\forall x \in \underline{\underline{B}}$

FOR $x \in A$, CLEAR: $\alpha|_x = (\alpha_1 - \alpha_2)|_x$

OTHERWISE: $x \in B_i$, $B_i \cap A \ni y$

OK SINCE: $(\alpha|_{B_i})|_y = 0$, AND ALSO

$|_y : H_m(\mathbb{R}^n|B_i) \to H_m(\mathbb{R}^n|y)$ IS ISO. □