RECALL:

M CLOSED CONNECTED. THEN:

M ORIENTABLE $\Rightarrow H_m(M) \cong \mathbb{Z}$

M NON-ORIENTABLE $\Rightarrow H_m(M) = 0$

POINCARÉ DUALITY:

M CLOSED CONN. ORIENT m-MFLD $\Rightarrow$

$\exists$ ISO $H^k(M) \longrightarrow H_{m-k}(M)$

WHAT'S THE MAP?

X = TOP SPACE, R = RING, $k \geq \ell$, DEFINE

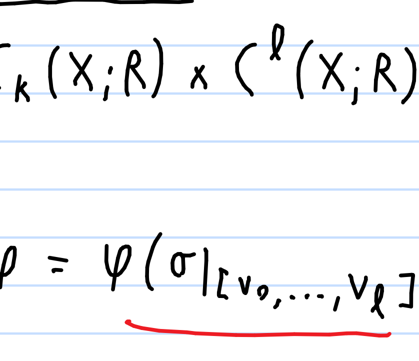
CAP PRODUCT:

$\cap : C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$

BY:

$\sigma \cap \varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\, \sigma|_{[v_\ell,\dots,v_k]}$

$k = 3$, $\ell = 1$



LEMMA: $\partial(\sigma \cup \varphi) = (-1)^\ell(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$

PROOF: COMPUTE $\square$

LEMMA $\Rightarrow \cap$ INDUCES A WELL-DEFINED, R-BILINEAR MAPS

$$\boxed{\cap : H_k(X;R) \times H^\ell(X;R) \longrightarrow H_{k+\ell}(X;R)}$$

FACT: THERE ARE ALSO RELATIVE VERSIONS

NATURALITY PROPERTY. $f : X \to Y$

$$\begin{array}{ccccc} H_k(X;R) & \times & H^\ell(X;R) & \longrightarrow & H_{k-\ell}(X) \\ \downarrow f_* & & \uparrow f^* & & \downarrow f_* \\ H_k(Y;R) & \times & H^\ell(Y;R) & \longrightarrow & H_{k-\ell}(Y) \end{array}$$

LEMMA $f : X \to Y$ CONT., $\alpha \in H_k(\underline{X};R)$, $\varphi \in H^\ell(\underline{Y};R)$. THEN:

$f_*(\alpha) \cap \varphi = f_*(\alpha \cap f_*(\varphi))$

THM (POINCARÉ DUALITY): M CLOSED CONN R-ORIENT m-MFLD, $[M] \in H_m(M;R)$ FUNDAMENTAL CLASS ($\Leftrightarrow$ GENERATOR) THEN $D : H^k(M;R) \to H_{m-k}(M;R)$ GIVEN BY $D(\varphi) = [M] \cap \varphi$ IS ISOMORPHISM

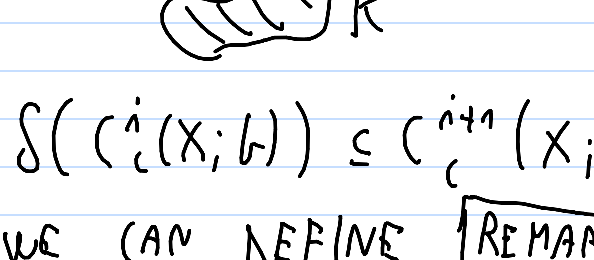
ROUGH IDEA OF PROOF (CFR: PROOF THAT $H_m(M;R) \cong R$):

PROVING SOME VERSION OF PD FOR LARGER AND LARGER OPEN SETS

SAID VERSION USES:

COHOMOLOGY WITH CPT SUPPORT.

DEFN: $C^i_c(X;G)$ IS THE SUBGP OF $C^i(X;G)$ OF ALL $\varphi : C_i(X;G) \to G$ S.T. $\exists\, K = K_\varphi \subseteq X$ CPT S.T. $\varphi(\sigma) = 0$ IF $im(\sigma) \subseteq X - K$



NOTE: $\delta(C^i_c(X;G)) \subseteq C^{i+1}_c(X;G)$

HENCE, WE CAN DEFINE

$H^*_c(X;G) := \frac{\ker \delta}{\operatorname{im} \delta}$

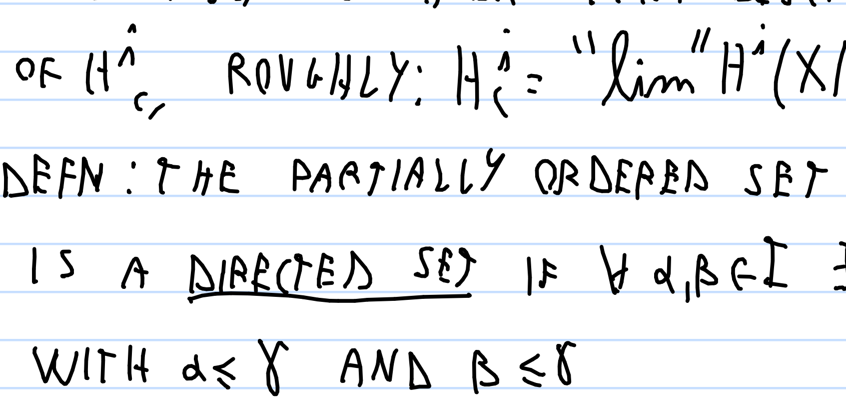
REMARK:

$C^i_c(X;G) = \bigcup_{K\, CPT} C^i(X, X-K; G)$

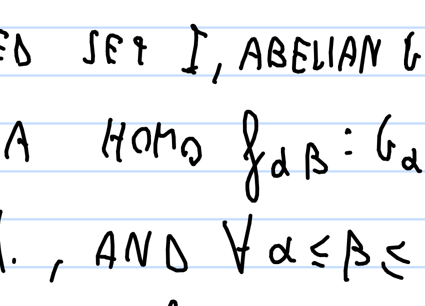
WE'LL ALSO USE A DIFFERENT DESCRIPTION OF H^i_c, ROUGHLY: $H^i_c = $ "$\lim$" $H^i(X|K)$

DEFN: THE PARTIALLY ORDERED SET I IS A DIRECTED SET IF $\forall\, \alpha, \beta \in I\ \exists\, \gamma \in I$ WITH $\alpha \leq \gamma$ AND $\beta \leq \gamma$

DEFN: A DIRECTED SYSTEM OF GPS IS A DIRECTED SET I, ABELIAN GPS G_α FOR $\alpha \in I$, $\forall\, \alpha \leq \beta$ A HOMO $f_{\alpha\beta} : G_\alpha \to G_\beta$, WITH $f_{\alpha\alpha} = id.$, AND $\forall\, \alpha \leq \beta \leq \gamma$ WE HAVE $f_{\alpha\gamma} = f_{\beta\gamma} \circ f_{\alpha\beta}$



DEFN: THE DIRECTED LIMIT OF A DIR SYS IS, AS A SET, $\coprod_\alpha G_\alpha / \sim$, WHERE $a \sim b$ IF $f_{\alpha\gamma}(a) = f_{\beta\gamma}(b)$ FOR SOME γ



THE DIR. LIM. IS A GROUP WITH THE "OBVIOUS" OPERATION

KEY POINT: ANY TWO EQUIV CLASSES HAVE REPR. IN SAME G_γ

NOTATION: $\varinjlim G_\alpha$

DIRECTED SYSTEM WE'RE INTERESTED IN:

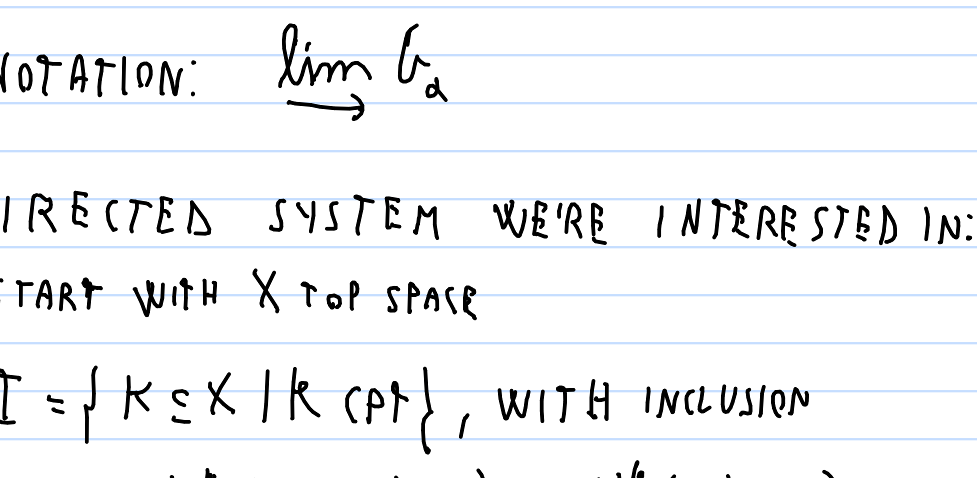
START WITH X TOP SPACE

$I = \{ K \subseteq X \mid K \text{ CPT} \}$, WITH INCLUSION

$G_K = H^k(X, X-K; G) =: H^k(X|K;G)$

MAPS: INDUCED BY INCLUSION

CONNECTIONS AMONG VARIOUS NOTIONS:



LEMMA: $H^i_c(X;G)$ IS NATURALLY ISOMORPHIC TO $\varinjlim H^i(X|K;G)$

SKETCH: JUST USE THE FOLLOWING.

ABSTRACT NON-SENSE FACT:

GIVEN A DIR. SYS. AND HOMOS $\psi : G_\alpha \to G$ WITH $\psi_\alpha = \psi_\beta \circ f_{\alpha\beta}$ FOR $\alpha \leq \beta$, $\exists$ HOMO $\psi : \varinjlim G_\alpha \to G$ GIVEN BY

$\psi([g]) = \psi_\alpha(g)$ IF $g \in G_\alpha$ $\square$

GENERAL POINCARÉ DUALITY.

M m-MFLD, R-ORIENTABLE. WE WANT $D : H^k_c(M;R) \to H_{m-k}(M;R)$

RECALL FROM LAST TIME: GIVEN R-ORIENT $x \mapsto \alpha_x \in H_m(M|x;R)$,

$\forall\, K \subseteq M$ CPT $\exists\, \mu_K \in H_m(M|K;R)$ WHOSE IMAGE IN $H_m(M|x;R)$ IS α_x

IN THIS SETTING, WE HAVE MAPS

$H^k(M|K;R) \longrightarrow H_{m-k}(M;R)$

$x \longmapsto \mu_K \cap x$

ABSTRACT NON-SENSE FACT GIVES

$D : H^k_c(M;R) \longrightarrow H_{m-k}(M;R)$

THM (MORE GENERAL PD)

M R-ORIENTED. THEN D AS ABOVE IS ISOMORPHISM.

THIS IS INDEED MORE GENERAL SINCE:

RMK: X CPT $\Rightarrow H^*_c(X;R) = H^*(X;R)$