RECALL:

M m-MFLD

$x \mapsto \alpha_x \in H_m(M|x;R)$ R-ORIENTATION

$\Rightarrow \forall K \subseteq M$ CPT $\exists \mu_K \in H_m(M|K;R)$

MAPPING TO α_x $\forall x \in K$

$H^i_c := \varinjlim H^i(X|K;R)$

$$D_K : H^i(X|K;R) \longrightarrow H_{m-i}(X;R)$$
$$\alpha \longmapsto \mu_K \frown \alpha$$

PASSING TO $\varinjlim$, THE D_K GIVE:

$$D_M : H^i_c(X) \longrightarrow H_{m-i}(X;R)$$

THM: D_M IS ISO.

WE OMIT R BELOW.

THE FOLLOWING ALLOW US TO PUT PIECES TOGETHER:

LEMMA: SUPPOSE $M = U \cup V$, OPEN. THEN $\exists$ A DIAGRAM THAT COMMUTES UP TO SIGN:

$$\begin{tikzcd}
\cdots \arrow[r] & H^i_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^i_c(U)\oplus H^i_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^i_c(M) \arrow[r] \arrow[d, "D_M"] & H^{i+1}_c(U\cap V) \arrow[d, "D_{U\cap V}"] \cdots \\
\cdots \arrow[r] & H_{m-i}(U\cap V) \arrow[r] & H_{m-i}(U)\oplus H_{m-i}(V) \arrow[r] & H_{m-i}(M) \arrow[r] & H_{m-i-1}(U\cap V)
\end{tikzcd}$$

HIGHLIGHTS OF PROOF:

- THE FIRST ROW IS OBTAINED AS $\varinjlim$ OF THE FOLLOWING M-V SEQUENCE, WHERE $K \subseteq U$ AND $L \subseteq V$ ARE COMPACT

$$\begin{tikzcd}
H^i(M|K\cap L) \arrow[r] \arrow[d, "\cong"] & H^i(M|K)\oplus H^i(M|L) \arrow[r] \arrow[d, "\cong"] & H^i(M|K\cup L) \\
H^i(U\cap V|K\cap L) & H^i(U|K)\oplus H^i(V|K) &
\end{tikzcd}$$

EXCISION

- HARD PART: "ALMOST" COMMUTATIVITY OF:

$$\begin{tikzcd}
H^i(M|K\cup L) \arrow[r, "\delta^*"] \arrow[d, "\mu_{K\cup L}\frown"] & H^{i+1}(M|K\cap L) \arrow[r] & H^{i+1}(U\cap V|K\cap L) \arrow[d, "\mu_{K\cap L}\frown"] \\
H_{m-i}(M) \arrow[rr, "\partial_*"] & & H_{m-i-1}(U\cap V)
\end{tikzcd}$$

CONNECTING HOMO

LESS CONFUSING VERSION:

$X = A \cup B$, OPEN $\quad \alpha \in H_m(X)$

THEN THE FOLLOWING COMMUTES UP TO SIGN:

$$\begin{tikzcd}
H^i(A\cap B) \arrow[r, "\delta^*"] \arrow[d, "\alpha\frown"] & im(\delta^*) \subseteq H^{i+1}(X) \arrow[d, "\alpha\frown"] \\
H_{m-i}(X) \arrow[r, "\partial_*"] & H_{m-i-1}(A\cap B)
\end{tikzcd}$$

LET $[\varphi] \in H^i(A\cap B)$, $\varphi = \varphi_A - \varphi_B$ FOR SOME

$\varphi_A \in C^i(A)$, $\varphi_B \in C^i(B)$

UNDER THE ISO $H^{i+1}(X) \cong H^{i+1}(C^*(A+B))$,

$\delta^*[\varphi] = \delta\varphi_A$

DUAL OF $\{A,B\}$-SMALL CHAINS

IF $\alpha = [c]$, FOR $c \in Z_m(A+B)$, THEN

FOLLOWING RED ARROW:

$[\varphi] \longmapsto [c \frown \delta\varphi_A]$

FOLLOWING BLUE ARROW:

$[\varphi] \mapsto [c\frown\varphi] \mapsto [\partial(c\frown\varphi_A)]$

$c = c\frown\varphi_A - c\frown\varphi_B$, THEN USE DEFN OF ∂_*

SUPPORT $\subseteq A$ $\quad$ SUPPORT $\subseteq B$

THOSE ARE SAME HOMOLOGY CLASS UP TO SIGN, SINCE:

$$\partial(c\frown\varphi_A) = (-1)^i(\underbrace{\partial c}_{=0}\frown\varphi_A - c\frown\delta\varphi_A)$$

PROOF OF THM:

WE USE 2 "STABILY PROPERTIES":

A) $M = U \cup V$, OPEN. IF THM HOLDS FOR U, V, AND $U \cap V$, THEN IT HOLDS FOR M.

THIS FOLLOW FROM LEMMA ABOVE + 5-LEMMA

B) $U_1 \subseteq U_2 \subseteq \ldots$ $\quad M = \bigcup_j U_j$. IF THM HOLDS FOR ALL U_j, THEN IT HOLDS FOR M

PROOF OF B IS LIMIT ARGUMENT:

$H^i_c(U_j)$ CAN BE TURNED INTO DIRECTED SYSTEM AND $\varinjlim H^i_c(U_j) \cong H^i_c(M)$

ALSO, THE D_{U_i} GIVE AN ISOMORPHISM

$\varinjlim H^i_c(U_j) \longrightarrow \varinjlim H_{m-i}(U_j)$, AND

THE LATTER IS $\cong H_{m-i}(U_j)$ (EXERCISE)

GIVEN A) AND B) WE CAN PROCEED AS FOLLOWS:

STEP 1: THM HOLDS FOR $\mathbb{R}^m$

FOR B_k: CLOSED BALL OF RADIUS k AROUND 0, WE HAVE

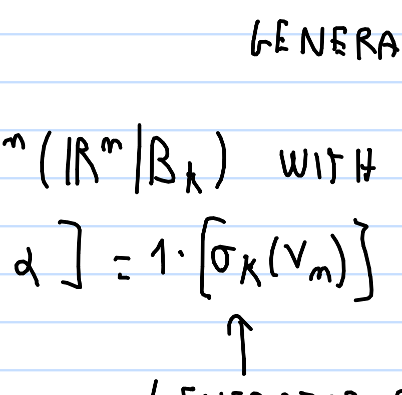
$$H^i_c(\mathbb{R}^m) \cong \varinjlim H^i(\mathbb{R}^m|K) = \varinjlim H^i(\mathbb{R}^m|B_k)$$

WE HAVE $H^i(\mathbb{R}^m|B_k) = \begin{cases} 0 & i \neq m \\ R & i = m \end{cases}$

HENCE, SAME HOLDS FOR $H^i_c(\mathbb{R}^m)$

$D_{\mathbb{R}^m} : H^i_c(\mathbb{R}^m) \longrightarrow H_{m-i}(\mathbb{R}^m)$ IS THE ISO BETWEEN 0s FOR $i \neq m$.

$i = m$: ONE CAN SHOW $\mu_{B_k} = [\sigma_k]$



$H^m(\mathbb{R}^m|B_k) \cong HOM(H_m(\mathbb{R}^m|B_k))$

GENERATED BY $[\sigma_k]$

FOR $\alpha \in H^m(\mathbb{R}^m|B_k)$ WITH $\alpha([\sigma_k]) = 1$,

$[\sigma_k \frown \alpha] = 1 \cdot [\sigma_k(v_m)]$

GENERATOR OF $H_0(\mathbb{R}^m)$ ✓

STEP 2: THM HOLDS FOR M AN OPEN SET OF $\mathbb{R}^m$

WRITE $M = \bigcup_j V_j$, V_j OPEN CONVEX

BY A), STEP 1, AND INDUCTION, THM HOLDS FOR

$U_j = V_1 \cup \ldots \cup V_j$. (NOTE: $U_j \cap V_{j+1} = (V_1 \cap V_{j+1}) \cup \ldots$)

CONVEX

BY B), THM HOLDS FOR M.

STEP 3: GENERAL CASE.

M 2ND COUNTABLE $\Rightarrow M = U_1 \cup U_2 \cup \ldots$

$U_i \cong \mathbb{R}^m$. PROCEED AS ABOVE. $\square$