GOAL: DEVELOP MORE "DUALITY THEORY", SO WE CAN PROVE:

THM 1. M CLOSED NON-ORIENTABLE m-MFLD THEN M DOES NOT EMBED IN S^{m+1}, OR $\mathbb{R}^{m+1}$ (E.G. $\mathbb{RP}^2$, KLEIN BOTTLE $\not\hookrightarrow \mathbb{R}^3$)

THM 1 WILL FOLLOW FROM:

THM 2: K CPT, LOCALLY CONTRACTIBLE SUBSPACE OF CLOSED ORIENTABLE m-MANIFOLD M. THEN $H_i(M, M-K; \mathbb{Z}) \cong H^{m-i}(K; \mathbb{Z})$

PROOF OF THM 2:

LET U OPEN NEIGH OF K. THERE'S A DIAGRAM:

$$\begin{tikzcd}
\cdots H_i(M-K) \arrow[r] & H_i(M) \arrow[r] & H_i(M,M-K) \arrow[r] & H_{i-1}(M-K) \cdots \\
 & & H_i(U,U-K) \arrow[u, "\cong"] & \\
H^{m-i}(M-K,U-K) \arrow[uu, "(1)"] & & & H^{m-i+1}(M-K,U-K) \arrow[uu] \\
\cdots H^{m-i}(M,U) \arrow[r] \arrow[u, "\cong"] & H^{m-i}(M) \arrow[r] \arrow[uuu] & H^{m-i}(U) \arrow[r] \arrow[uu, "(2)"] & H^{m-i+1}(M,U) \cdots \arrow[u, "\cong"]
\end{tikzcd}$$

— = EXCISION

— = MAP FROM POINCARÉ DUALITY

$H^j(M-K, U-K) = H^j(M-K \mid U^c)$

ONE CAN CHECK: DIAGRAM COMMUTES UP TO SIGNS (NOTE: MUCH EASIER THAN IN PROOF OF PD!)

TAKING $\varinjlim$ OVER U DECREASING TO K,

(1) BECOMES THE PD ISO $H_c^{m-i}(M-K) \to H_i(M-K)$

BY 5-LEMMA, (2) BECOMES ISO

$\varinjlim H^{m-i}(U) \longrightarrow H_i(M, M-K)$

FACT THAT WE ACCEPT FOR NOW:

$$\varinjlim H^{m-i}(U) \cong H^{m-i}(K) \qquad \square$$

SMALL DIGRESSION:

LEMMA: M NON-CPT, CONNECTED m-MFLD. THEN $H_m(M;\mathbb{Z}) = 0$

PROOF: $H_m(M;\mathbb{Z}) \cong H_c^0(M;\mathbb{Z})$ BY PD

FROM EX. SHEET: $H_c^0(M;\mathbb{Z}) = 0$ FOR M NON-CPT $\square$

COR (OF THM 2): K CPT, LOC. CONTR., $\neq \emptyset$ PROPER SUBSPACE OF S^m, THEN

$$\tilde{H}_i(S^m - K; \mathbb{Z}) \cong \tilde{H}^{m-i-1}(K; \mathbb{Z})$$

ALEXANDER DUALITY

PROOF: LES FOR PAIR (S^m, S^m-K) GIVES ISO $\tilde{H}_i(S^m-K) \cong H_{i+1}(S^m, S^m-K)$ FOR ALL i EXCEPT $i = m-1$. WE HAVE:

$$0 \to \tilde{H}_m(S^m) \longrightarrow \tilde{H}_m(S^m, S^m-K) \to \tilde{H}_{m-1}(S^m-K) \to 0$$

$\overset{\shortparallel}{H_m(S^m-K)}$

BY PREVIOUS LEMMA APPLIED TO CONN COMP.

THE SEQUENCE SPLITS: FOR $K' = K - \{p\}$, WITH $S^m = \mathbb{R}^m \cup \{p\}$, WE CAN CONSTRUCT A SECTION USING:

$$\begin{tikzcd}
 & \tilde{H}_m(S^m, S^m-K) \arrow[r] & \tilde{H}_{m-1}(S^m-K) & \\
\overset{0}{\overset{\shortparallel}{\tilde{H}_m(\mathbb{R}^m)}} \arrow[r] & \tilde{H}_m(\mathbb{R}^m, \mathbb{R}^m-K') \arrow[r, "\cong"] \arrow[u] & \tilde{H}_{m-1}(\mathbb{R}^m-K') \arrow[r] \arrow[u, equal] & \overset{0}{\overset{\shortparallel}{\tilde{H}_{m-1}(\mathbb{R}^m)}}
\end{tikzcd}$$

SO: $\tilde{H}_m(S^m, S^m-K) \cong \tilde{H}_{m-1}(S^m-K) \oplus \mathbb{Z}$

$\qquad \overset{\cong}{\tilde{H}^0(K)} \qquad \cong \tilde{H}^0(K) \oplus \mathbb{Z}$

$\Rightarrow \tilde{H}_{m-1}(S^m-K) \cong \tilde{H}^0(K) \qquad \square$

PROOF OF THM 1:

IF M EMBEDS INTO S^{m+1}, THEN

$$\tilde{H}_0(S^{m+1} - M) \cong \tilde{H}^m(M)$$

M NON-ORIENTABLE $\Rightarrow \tilde{H}^m(M) \underset{\text{UCT}}{\cong} \text{TORSION}(H_{m-1}(M)) \neq 0$

UCT FOR $H^m(M; \mathbb{Z}/2)$

BUT $\tilde{H}_0$ IS ALWAYS TORSION-FREE ↯

LEFT: $U \searrow K \Rightarrow \varinjlim H^j(U) = H^j(K)$?

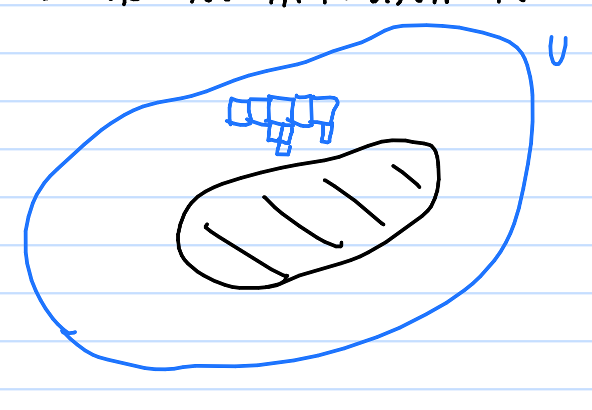
ISSUE: K NEED NOT BE HOMOTOPY EQUIV. TO SMALL NEIGHBORHOODS

SOLUTION: THEORY OF NEIGHBORHOOD RETRACTS

PROP: $K \subseteq \mathbb{R}^n$ CPT + LOCALLY CONTRACTIBLE $\Rightarrow K$ IS A RETRACT OF SOME NEIGHBORHOOD

$A \subseteq X$ IS RETRACT IF $\exists\, r: X \to A$ CONT. WITH $r|_A = id$

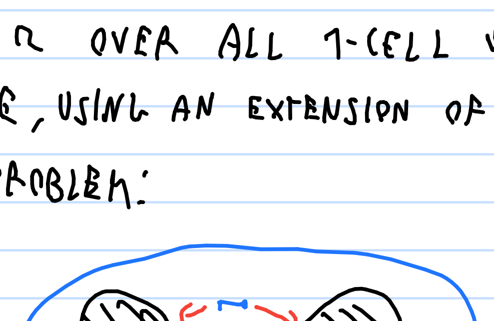
SKETCH OF PROOF: START WITH ANY NEIGH U. FIND CW-STRUCTURE ON $X = U - K$ S.T. SIZE OF CELLS GO TO 0 AS YOU APPROACH K



DEFINE $r: X^{(0)} \to K$ BY MAPPING v TO ANY CLOSEST POINT IN K

EXTEND r OVER ALL 1-CELL WHERE IT'S POSSIBLE, USING AN EXTENSION OF ALMOST MINIMAL DIAMETER.

SAMPLE PROBLEM:

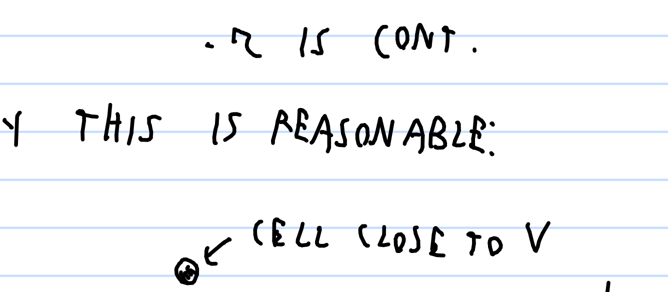


REPEAT OVER ALL SKELETA.

IN THE END: SOME $X' \subseteq X$, SOME $r: X' \to K$, THAT CAN BE EXTENDED TO $r: X' \to K$

TO SHOW: • X' IS NEIGH
• r IS CONT.

WHY THIS IS REASONABLE:



$\Rightarrow$ 0-SKELETON OF Q-CELL MAPS TO V. "□"

X TOP SPACE IS EUCLIDEAN NEIGHBORHOOD RETRACT (ENR) IF $\exists$ EMBEDDING $i: X \to \mathbb{R}^k$

PROP: M CPT MFLD $\Rightarrow M$ IS ENR

PROOF: M IS COVERED BY FINITELY MANY CLOSED m-BALLS B_i. LET $f_i: M \to S^m$ COLLAPSE $M - \mathring{B}_i$ TO A POINT, WITH $f_i|_{\mathring{B}_i}$ AN EMBEDDING.

THE f_i GIVE $M \to (S^m)^k$ EMBEDS IN SOME $\mathbb{R}^n$ $\square$

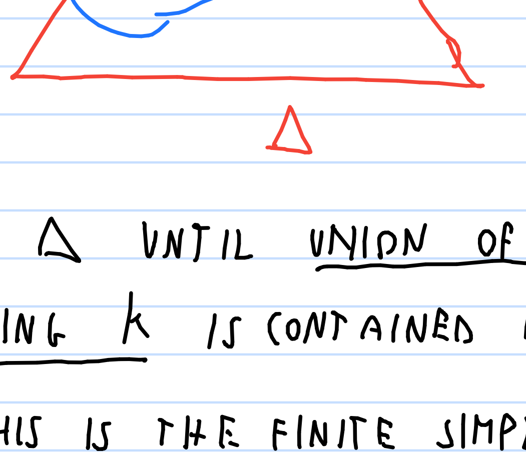
EXERCISE: M CLOSED MFLD, $K \subseteq M$ LOCALLY CONTR AND CPT. THEN $\varinjlim H^j(U) \cong H^j(K)$, AS $U \searrow K$.

ANOTHER NICE THING ABOUT ENR:

PROP: X CPT ENR $\Rightarrow X$ IS A RETRACT OF A FINITE SIMPLICIAL CPLX.
IN PART, ALL $H_i(X)$ AND $\pi_1(X, *)$ ARE FIN. GEN.

PROOF: SAY X IS RETRACT OF U.

X CPT $\Rightarrow X$ CONTAINED IN (IMAGE OF) A SPLX



SUBDIVIDE Δ UNTIL UNION OF SPLICES INTERSECTING K IS CONTAINED IN U

THIS IS THE FINITE SIMPL CPLX $\square$