[CONTINUE FROM LEC. 19]

AND WHILE WE'RE AT IT:

LEMMA: K FINITE CW-CPLX $\Rightarrow$ K IS ENR

PROOF: ENOUGH TO SHOW THAT K EMBEDS IN $\mathbb{R}^m$

INDUCTION ON NUMBER OF CELLS: SUPPOSE $K = A \cup_g D^k$, WITH $A \subseteq \mathbb{R}^m$

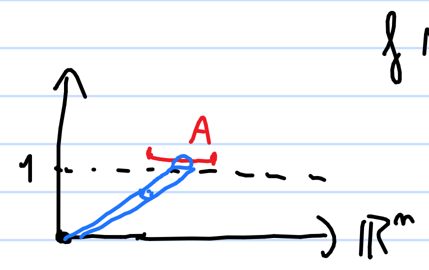
THEN WE CAN CONSTRUCT THE EMBEDDING:

$$
\begin{aligned}
K &\longrightarrow \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R} \\
x \in A &\longmapsto (x,\ 0,\ 1) \\
t \cdot v &\longmapsto (t \cdot g(v),\ v,\ t) \quad \square
\end{aligned}
$$

$t \in [0,1], v \in S^{k-1}$

NEEDED BECAUSE g NEED NOT BE INJ.

IF INJ:



RMK: IN FACT, ONE CAN MAKE IFF STATEMENTS:

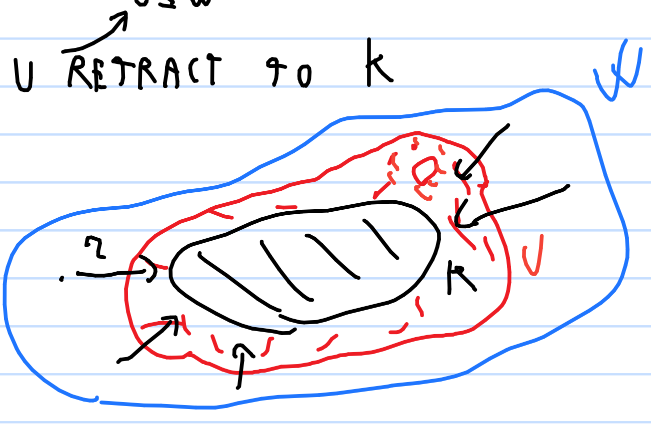
- $K \subseteq \mathbb{R}^m$ CPT IS RETRACT OF A NEIGH $\Leftrightarrow$ IT'S SEMI-LOC CONTRACTIBLE
- K CPT. THEN K IS ENR $\Leftrightarrow$ K IS RETRACT OF FINITE SIMPL CPLX

EXERCISE: M CLOSED MFLD, $K \subseteq M$ LOCALLY CONTR AND CPT. THEN $\varinjlim H^s(U) \cong H^s(K)$ AS $U \searrow K$.

PROOF: WLOG, $M \subseteq \mathbb{R}^m$. THEN K IS RETRACT OF A NEIGH IN $\mathbb{R}^m$ $\Rightarrow$ K IS A RETRACT OF A NEIGH W IN M

$\varinjlim H^s(U) = \varinjlim_{U \subseteq W} H^s(U)$

ALL SUCH U RETRACT TO K



$r: X \to A$ RETRACTION $\Rightarrow$ $i^*: H^s(X) \to H^s(A)$ IS SURJ.

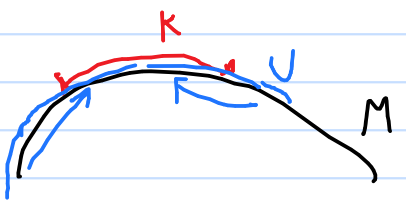
HENCE, NATURAL MAP $\varinjlim H^s(U) \longrightarrow H^s(K)$ IS SURJ.

FOR INJECTIVITY, WE SHOW:

$\alpha \in H^s(U)$ S.T. $\alpha|_K = 0$. THEN $\alpha|_V = 0$ FOR SOME NEIGH $V \subseteq U$ OF K

NOTATION: $\alpha|_Y = i^*(\alpha) \in H^s(Y)$

TO SHOW THIS, IN $\mathbb{R}^m$ $\mathrm{id}: U \to U$ IS HOMOTOPIC TO r, WITH HOMOTOPY H STATIONARY ON K



FOR V SUFF SMALL, $H(V \times [0,1]) \subseteq$ NEIGH OF M IN $\mathbb{R}^m$ THAT RETRACTS TO M

$\Rightarrow$ FOR V EVEN SMALLER, $i: V \to U$ IS HOMOTOPIC IN U TO $r: V \to K$

(MEANING $i: V \to U$ AND $i \circ r: V \to U$ ARE HOMOTOPIC)

SO WE HAVE:

$$
\begin{tikzcd}
H^s(U) \arrow[r, "i^*"] \arrow[rr, bend right, dashed, "i^*"'] & H^s(K) \arrow[r, "r^*"] & H^s(V)
\end{tikzcd}
$$

AND $\alpha|_V = r^*(\underbrace{\alpha|_K}_{=0}) = 0$ $\quad \square$