GROUP (CO)HOMOLOGY

G GROUP. HOW CAN ONE DEFINE $H_*(G)$?

ALGEBRAIC APPROACH:

$$G \rightsquigarrow \text{CHAIN CPLX} \rightsquigarrow H_*$$

TOPOLOGICAL APPROACH

ATTEMPT 1: $H_n(G) := H_n(X)$, X SOME TOP SPACE WITH $\pi_1(X) \cong G$

THIS DOESN'T WORK:

ISSUE 1: NOT WELL-DEFINED, E.G.

$$H_2(S^2) \ncong H_2(\{pt\})$$

ISSUE 2: LACK OF FUNCTORIALITY

NOTE: SOLVING ISSUE 2 SHOULD SOLVE ISSUE 1

KEY NOTION: X TOP SPACE IS <u>ASPHERICAL</u> IF $\forall n \neq 1$ $\forall f: S^n \to X$ CONT. $\exists$ EXTENSION $\bar{f}: D^{n+1} \to X$

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, hook] & X \\ D^{n+1} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

RMK: • TAKING $n=0$, WE SEE <u>ASPHERICAL</u> $\Rightarrow$ PATH-CONN.

• SAME AS SAYING $\pi_n(X) = 0$ $\forall n \neq 1$

• CONTRACTIBLE $\Rightarrow$ ASPHERICAL

LEMMA/EX: $p: \hat{X} \to X$ COVER, $\hat{X}$ AND X PATH-CONN. THEN X IS ASPHERICAL $\iff$ $\hat{X}$ IS ASPHERICAL

FOR ASPHERICAL CW-CPLXS, ISSUE 2 DOES NOT ARISE

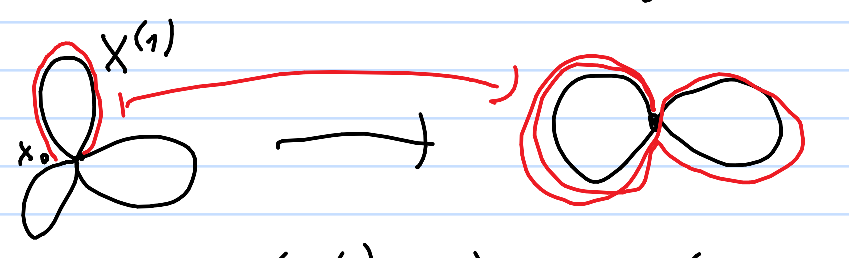
THM: X, Y CW-CPLXS. X CONN, Y ASPHERICAL. THEN $\forall g: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ HOMO $\exists f: (X, x_0) \to (Y, y_0)$ CONT S.T. $f_* = g$. MOREOVER, f IS UNIQUE UP TO HOMOTOPY REL BASEPOINTS.

SKETCH, FOR EXISTENCE.

SAY $X^{(0)} = \{x_0\}$, $Y^{(0)} = y_0$

WE DEFINE f INDUCTIVELY ON SKELETA:

- $f(x_0) = y_0$
- σ 1-CELL $\rightsquigarrow$ $f(\sigma)$ = LOOP IN $Y^{(1)}$ REPRESENTING $g([\sigma])$



NOTE: $i_*: \pi_1(Y^{(1)}, y_0) \to \pi_1(Y, y_0)$ IS SURJ (TRUE $\forall$ CW-CPLXS)

- EXTENDING TO $X^{(2)}$

σ 2-CELL $\Rightarrow$ $[\partial\sigma] = 0$ IN $\pi_1(X, x_0)$

$\Rightarrow$ $[f(\partial\sigma)] = 0$ IN $\pi_1(Y, y_0)$

$\Rightarrow$ f CAN BE EXTENDED ACROSS σ

NOTE: $i_*: \pi_1(Y^{(2)}, y_0) \to \pi_1(Y, y_0)$ IS ISO (TRUE $\forall$ CW-CPLXS) $\rightsquigarrow$ CAN ENSURE $f(X^{(2)}) \subseteq Y^{(2)}$

- EXTENDING TO HIGHER SKELETA

σ $(n+1)$-CELL $\Rightarrow$ "$f|_{\partial\sigma}$" CAN BE EXTENDED TO σ (BY ASPHERICITY) WITH IMAGE IN $Y^{(n+1)}$ (BY CELLULAR APPROXIMATION) $\square$

THIS MOTIVATES:

DEFN. G GROUP. A <u>$K(G,1)$ SPACE</u> IS AN ASPHERICAL CW-CPLX X S.T. $\pi_1(X) \cong G$.

COR OF THM: GIVEN G, ANY TWO $K(G,1)$ SPACES ARE HOMOTOPY EQUIVALENT.

DEFN: G GROUP, X $K(G,1)$ SPACE.

$$H_n(G) := H_n(X)$$
$$H^n(G) := H^n(X)$$

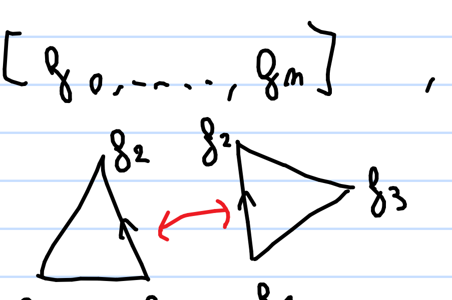
THIS IS WELL-POSED SINCE:

PROP: ANY GROUP G HAS A $K(G,1)$ SPACE

SKETCH: WE CONSTRUCT A CONTRACTIBLE CW-CPLX EG, WITH A G-ACTION (PERMUTING CELLS) AND SO THAT $EG \to EG/G = BG$ IS COVER

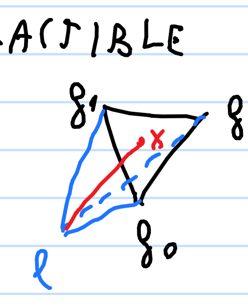
COVERING THEORY $\Rightarrow$ $\pi_1(BG) \cong G$, SO BG IS $K(G,1)$ SPACE.

IN FACT, EG IS Δ-CPLX WITH SIMPLICES $[g_0, \ldots, g_n]$, $g_i \in G$



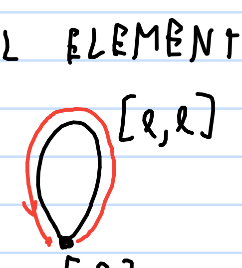
CLAIM: EG IS CONTRACTIBLE

CONTRACTION:



$\square$

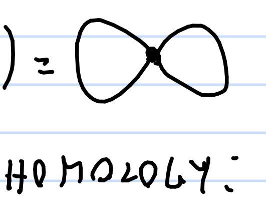
RMK: • e TRIVIAL ELEMENT, THEN



RMK: • THE BG AS CONSTRUCTED ABOVE HAS A RATHER EXPLICIT Δ-CPLX STRUCTURE $\rightsquigarrow$ ONE CAN USE IT TO GIVE ALGEBRAIC DESCRIPTION OF $H_*(G)$, $H^*(G)$

EXAMPLES OF $K(G,1)$:

"=" MEANS "CAN BE TAKEN TO BE"

- $K(\mathbb{Z}, 1) = S^1$
- $K(G \times H, 1) = K(G,1) \times K(H,1)$
- $K(\mathbb{Z}/_2, 1) = \mathbb{RP}^\infty$
- $K(\text{FREE GP ON 2 GEN}, 1) =$ 

(VERY) LOW-DEGREE (CO)-HOMOLOGY:

- $H_0(G) = H^0(G) = \mathbb{Z}$
- $H_1(G) = G^{ab}$
- $H^1(G) = \mathrm{HOM}(G; \mathbb{Z})$