CENTRAL EXTENSIONS

A CENTRAL EXTENSION OF G BY $\mathbb{Z}$ IS A SES

$$1 \to \mathbb{Z} \xrightarrow{i} E \xrightarrow{\jmath} G \to 1$$

WHERE $i(\mathbb{Z}) < Z(E)$ ← CENTRE OF E

EXAMPLES:

- $E = G \times \mathbb{Z}$
- $1 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/_2 \to 1$
- $H = \langle x, y, z \mid [x,z], [y,z], [x,y]z^{-1} \rangle$

$$\cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$ ← HEISENBERG GROUP

THERE'S A CE:

$$1 \to \mathbb{Z} \to H \to \mathbb{Z}^2 \to 1$$

- $1 \to \mathbb{Z} \to \pi_1(T^1 \Sigma_g) \to \pi_1(\Sigma_g) \to 1$

$\Sigma_g =$ 

EQUIVALENCE OF CE IS $\varphi : E \to E'$ S.T.

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z} & \to & E & \to & G & \to & 1 \\ & & \| & & \downarrow \varphi & & \| & & \\ 1 & \to & \mathbb{Z} & \to & E' & \to & G & \to & 1 \end{array}$$

NOTE: 5-LEMMA $\Rightarrow \varphi$ IS ISO

DEFN: A CE IS TRIVIAL IF IT'S EQUIVALENT TO $G \times \mathbb{Z}$

LEMMA/EX: A CE IS TRIVIAL $\Leftrightarrow \exists f : G \to E$ HOMO S.T. $\jmath \circ f = id$

$$\begin{tikzcd} E \arrow[r] & G \arrow[l, dashed, bend left, "f"] \end{tikzcd}$$

THM: GIVEN G, THERE IS A BIJECTION

$$\{CE\}/_{EQUIVALENCE} \longleftrightarrow H^2(G; \mathbb{Z})$$

E.G.: EVERY CE OF $\mathbb{Z}$ IS TRIVIAL, WHILE $\mathbb{Z}/_2$ HAS 2 CE / EQUIVALENCE

ASSIGNING A COHOM CLASS TO A CE:

$$H^2(G) = \frac{\{ f : G^3 \to \mathbb{Z} \mid G\text{-INVARIANT}, \delta f = 0 \}}{\{ \delta\varphi \mid \varphi : G^2 \to \mathbb{Z}, G\text{-INVARIANT} \}}$$

GIVEN THE CE

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & E \arrow[r, "\jmath"] & G \arrow[r] \arrow[l, dashed, bend left, "s"] & 1 \end{tikzcd}$$

CHOOSE A SET-THEORETIC SECTION $s : G \to E$, THAT IS $\jmath \circ s = id$

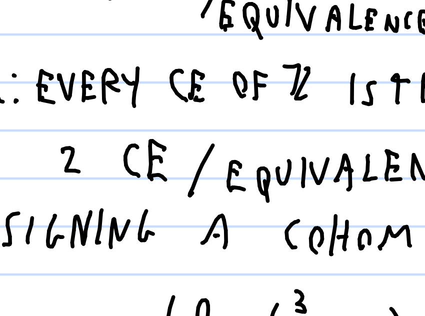
RMK: FOR $g_0, g_1 \in G$, WE HAVE

$$s(g_0)\, s(g_1)\, s(g_0 g_1)^{-1} \in \ker(\jmath) = im(i) \cong \mathbb{Z}$$

ALSO, IF s WAS HOMO, WE'D HAVE $= 0$.

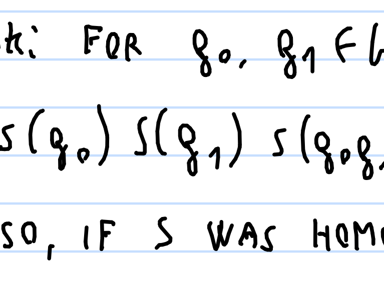
SET:

$$f(g_0, g_1, g_2) = s(g_0^{-1} g_1)\, s(g_1^{-1} g_2)\, s(g_0^{-1} g_2)^{-1}$$



$\alpha_E = [f] \in H^2(G)$ IS THE COHOM CLASS ASSOCIATED TO THE CE

IDEA WHY $\delta f = 0$:



$$\delta f(g_0, \dots, g_3) = \sum (-1)^i f(g_0, \dots, \hat{g}_i, \dots g_3)$$

$$= \widetilde{\prod} f(\dots)^{\pm 1}$$

↑ IN E ↖ THESE TERMS COMMUTE (BUT THE $s(\dots)$ MIGHT NOT!)

SAY $g_0 = 1$

THE TERM $s(g_1)$ APPEARS TWICE:

- $\left(s(g_1)\, s(g_1^{-1} g_2)\, s(g_2)^{-1} \right)^{-1} = \dots\dots s(g_1)^{-1}$
- $s(g_1)\, s(g_1^{-1} g_3)\, s(g_3)^{-1}$

$\rightsquigarrow$ CAN REARRANGE THE TERMS SO THAT THE $s(g_1)^{\pm 1}$ CANCEL OUT.

THERE IS TOPOLOGY BEHIND THESE...

$$\left\{ \begin{array}{c} CE \text{ OF } G \\ \text{BY } \mathbb{Z} \end{array} \right\} \longleftrightarrow \left\{ \pi_1(S^1\text{-BUNDLE OVER } K(G,1)) \right\}$$

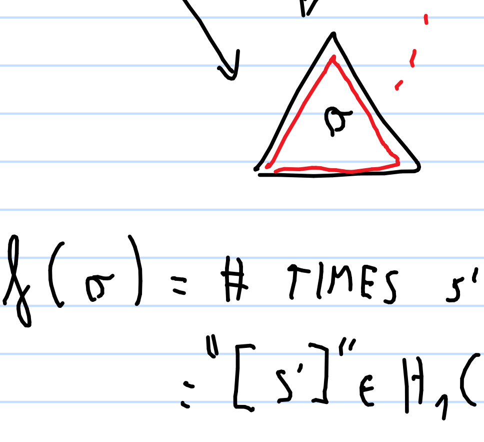
ROUGHLY: AN S^1-BUNDLE OVER A Δ-COMPLEX K IS OBTAINED GLUING COPIES OF $\sigma \times S^1$, σ SPLX OF K

E.G.:
- $K \times S^1$
- KLEIN BOTTLE, FOR $K = S^1$

COHOM CLASSE ASSOCIATED TO S^1-BUNDLE: OBSTRUCTION TO EXTENDING SECTION FROM $K^{(1)}$ TO $K^{(2)}$

SAY YOU HAVE BUNDLE $E \to K$, AND SECTION $s : K^{(1)} \to E$

BE EXTENDED ACROSS 2-SPLX σ?



$\rightsquigarrow s' : \partial\sigma \to S^1$

$$f(\sigma) = \#\text{ TIMES } s' \text{ WRAPS AROUND } S^1 =: \text{"}[s']\text{"} \in H_1(S^1; \mathbb{Z}) \cong \mathbb{Z}$$

$\rightsquigarrow [f] \in H^2(K)$.

CURIOSITY:

LEMMA: $1 \to \mathbb{Z} \to E \xrightarrow{\jmath} G \to 1$ CE.

α_E HAS FINITE ORDER $\Leftrightarrow \exists\, H < G$ FIN. INDEX S.T.

$$1 \to \mathbb{Z} \to \jmath^{-1}(H) \to H \to 1$$

IS TRIVIAL

AN APPLICATION OF $H_*(G)$:

THM: LET $f : \mathbb{R}^n \circlearrowleft$ BE HOMEO, $f \circ f = id$. THEN f HAS A FIXED POINT

SKETCH: WE HAVE A $\mathbb{Z}/_2$-ACTION ON $\mathbb{R}^n$.

CAN CHECK: $\mathbb{R}^n \to \mathbb{R}^n/_{\mathbb{Z}/_2} =: X$ IS COVER

FACT: X IS HOMOT. EQUIV TO CW-CPLX

$$\Rightarrow H_n(X) \cong H_n(K(\mathbb{Z}/_2, 1)) = H_n(\mathbb{Z}/_2)$$

THIS IS NON-0 FOR ALL ODD n.

HOWEVER, X n-MFLD $\Rightarrow H_k(X) = 0\ \forall k > n$ ∎

SIMILARLY:

THM: LET G BE FINITE, NON-TRIVIAL GROUP THEN THERE IS NO FREE G-ACTION ON $\mathbb{R}^n$

THIS USES:

EXERCISE*: COMPUTE $H_*(\mathbb{Z}/_m)$.