CUP PRODUCT

$\cup : H^k(X) \times H^l(X) \longrightarrow H^{k+l}(X)$

APPROACH 1 (THE NATURAL ONE)

FIRST STEP:

$H^k(X) \times H^l(X) \xrightarrow{\times} H^{k+l}(X \times X)$ CROSS PRODUCT

EASY FOR CELLULAR COHOMOLOGY

$$(\varphi \times \psi)(\underset{\substack{\uparrow \\ \text{CELL IN} \\ X \times X}}{\sigma}) = \begin{cases} \varphi(\sigma_1)\psi(\sigma_2) & \text{IF } \sigma = \underset{\substack{\uparrow \\ k\text{-CELL}}}{\sigma_1} \times \underset{\substack{\uparrow \\ l\text{-CELL}}}{\sigma_2} \\ 0 & \text{IF NOT} \end{cases}$$

SECOND STEP: WE ALSO HAVE $f : X \longrightarrow X \times X$, $x \longmapsto (x,x)$

SO JUST DEFINE $\varphi \cup \psi = f^*(\varphi \times \psi)$

CONTRAVARIANCE IS CRUCIAL!

$\rightarrow$ $\exists$ CROSS PRODUCT FOR H_*, BUT THEN THERE'S NO NATURAL "INTERESTING" MAP $X \times X \to X$

APPROACH 2: MAGIC FORMULA

(FROM $\cup$ ONE CAN RECONSTRUCT $\times$

WE TAKE APPROACH 2 BECAUSE $\times$ IS HARD TO CONSTRUCT.

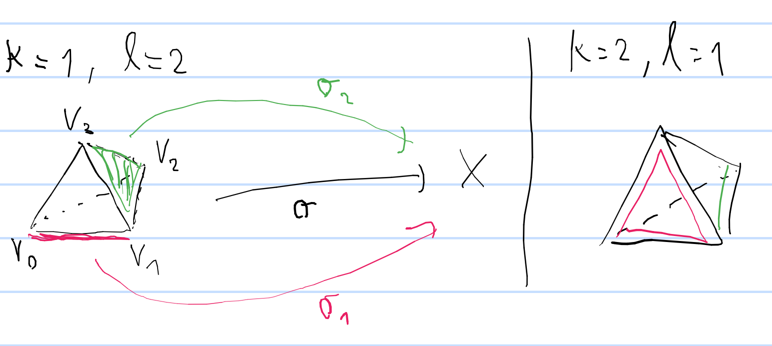
FIX R RING WITH 1 ($\mathbb{Z}$, $\mathbb{Z}/_n$, $\mathbb{Q}$, OR $\mathbb{R}$)

DEFN: FOR $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$ DEFINE THE <u>CUP PRODUCT</u> $\varphi \cup \psi \in C^{k+l}(X;R)$ BY

$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,v_{k+l}]}),$$

$\sigma : \Delta^{k+l} \longrightarrow X$, Δ^{k+l} = STANDARD SPLX $= [v_0, \dots, v_{k+l}]$

E.G. $k=1$, $l=2$ | $k=2$, $l=1$



$$\varphi \cup \psi(\sigma) = \varphi(\sigma_1)\,\psi(\sigma_2)$$

DOES IT PASS TO $H^k \times H^l$? YES!

RELATION WITH δ:

LEMMA: $\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^k \varphi \cup \delta\psi$

RECALL: $\delta\varphi(\sigma) = \varphi(\partial\sigma)$

PROOF: $(\delta\varphi \cup \psi)(\underset{\substack{\uparrow \\ k+l+1\text{-SPLX}}}{\sigma}) = (\delta\varphi)(\sigma|_{[v_0,\dots,v_{k+1}]})\,\psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$

$$= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\,\psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]}) \quad (1)$$

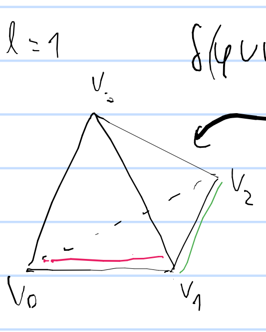
$$(\varphi \cup \delta\psi)(\sigma) = \sum_{i=0}^{l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,\hat{v}_i,\dots,v_{k+l+1}]})$$

$$(-1)^k \cdot \quad = \sum_{i=k}^{k+l+1} (-1)^i \quad (2)$$

LAST TERM OF 1 = FIRST TERM OF 2

REMAINING TERMS $= \delta(\varphi \cup \psi)(\sigma)$ $\square$

$k=l=1$ $\quad \delta(\varphi \cup \psi)(\sigma) = \sum$ FACES OF $\varphi \cup \psi(\sigma)$

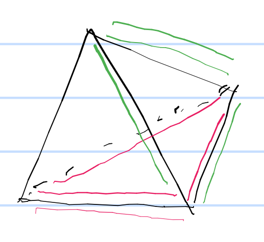


ONE OF THE TERMS

— = EVALUATE φ

— = EVALUATE ψ

SUM =



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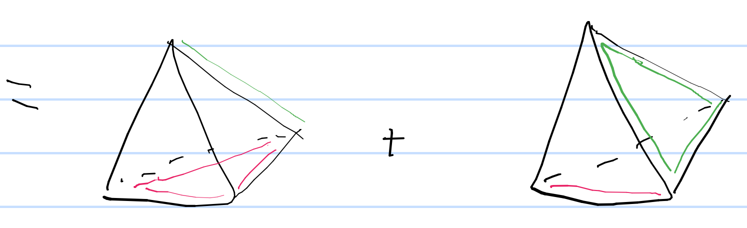


LEMMA $\Rightarrow$ $\cup$ IS WELL-DEFINED IN H^*:

THM: $\cup : H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R)$ GIVEN BY $[\varphi] \cup [\psi]$ IS WELL-DEFINED.

MOREOVER:

- $\cup$ IS ASSOCIATIVE
  $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$
- $\cup$ IS DISTRIBUTIVE
  $\alpha \cup (\beta + \gamma) = \alpha \cup \beta + \alpha \cup \gamma$, $(\alpha + \beta) \cup \gamma = \alpha \cup \gamma + \beta \cup \gamma$
- (IDENTITY) IF R HAS 1. DENOTE $1 \in H^0(X;R)$ THE CLASS OF CONST MAP = 1 ($\varphi(\sigma) = 1$ $\forall$ SING 0-SPLX)

  THEN $\alpha \cup 1 = 1 \cup \alpha = \alpha$

PROOF

LEMMA $\Rightarrow$ 1) φ, ψ COCYCLES (I.E. $\in \ker \delta$) $\Rightarrow$ $\varphi \cup \psi$ COCYCLE

2) φ COCYCLE, ψ $\underset{\substack{\uparrow \\ \in \operatorname{Im} \delta}}{\text{COBOUNDARY}}$ $\Rightarrow$ $\varphi \cup \psi$ COBOUNDARY

3) φ COBOUNDARY, ψ COCYCLE $\Rightarrow$ $\varphi \cup \psi$ COBOUNDARY.

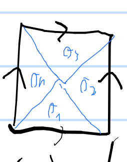
1) + 2) + 3) $\Rightarrow$ $\cup$ IS WELL-DEFINED IN H^*

$\cup$ IS ASSOCIATIVE AND DISTR. IN $C^* \Rightarrow$ IT IS ALSO IN H^*

IDENTITY: DIRECT CHECK $\square$

NEXT TIME

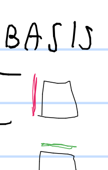
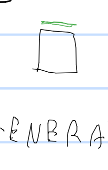
EXAMPLE: TORUS

T: 

$\leftarrow$ MAKE IT A Δ-CPLX + USE H^Δ_n

RECALL: $H_n(T) = \begin{cases} \mathbb{Z} & n = 0,2 \\ \mathbb{Z}^2 & n = 1 \\ 0 & \text{OTHERWISE} \end{cases}$

FACTS: 1) $H_1(T)$ HAS BASIS $\{a_1, b_1\}$ WHERE:

- $a_1 = [$  $]$
- $b_1 = [$  $]$

2) $H_2(T)$ IS GENERATED BY $[\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4]$

ALL H_i ARE FREE $\Rightarrow$ $H^n(T;\mathbb{Z}) \cong \operatorname{HOM}(H_n(T);\mathbb{Z}) \cong H_n(T)$ $\forall n$

THE ISO IS $h([\varphi])([c]) = [\varphi(c)]$

LET US WRITE $\alpha([c])$ FOR $h(\alpha)([c])$

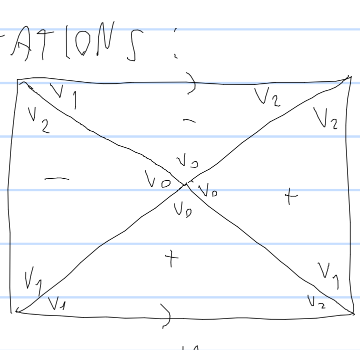
LET $\alpha_i \in H^1(T;\mathbb{Z})$ BE SO THAT $\alpha_i(a_i) = 1$, $\alpha_i(b_1)$

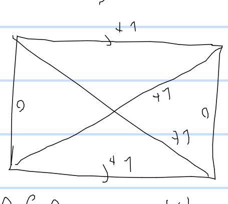
β_i S.T. $\beta_i(a_i) = 0$, $\beta_i(b_i) = 1$

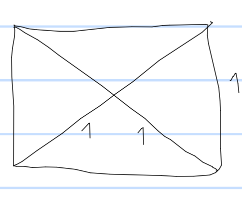
NEED: REPRESENTATIVE φ_1 OF α_1 IN C^1.

BY THM, WE CAN JUST PICK A $\underset{\substack{\uparrow \\ \text{STILL NEEDS TO BE A COCYCLE!}}}{\text{REP}}$ IN Δ^1_1

ORIENTATIONS:



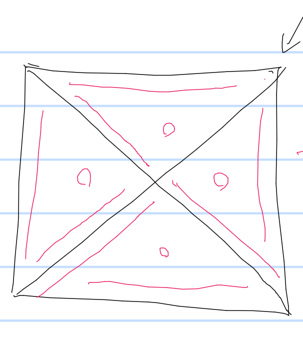
φ_1 

ψ_1 

SIMILARLY, REP ψ_1 FOR β_1

COMPUTATIONS:

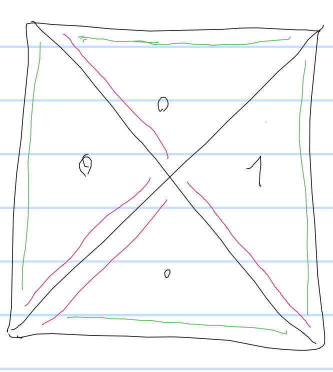
- $(\alpha_1 \cup \alpha_1)(c_2) = 0$



AT LEAST ONE RED EDGE IN EACH Δ HAS $\varphi_1 = 0$

— = EVALUATE α

- $(\alpha_1 \cup \beta_1)(c_2) = +1$



— = EVALUATE α_1

— = EVALUATE β_1

- $(\beta_1 \cup \alpha_1)(c_2) = -1$

HENCE: LET γ BE GENERATOR OF $H^2(T;\mathbb{Z})$ S.T. $\gamma(c_2) = 1$. THEN

$$\begin{cases} \alpha_1 \cup \alpha_1 = 0 \\ \beta_1 \cup \beta_1 = 0 \\ \alpha_1 \cup \beta_1 = -(\beta_1 \cup \alpha_1) = \gamma \end{cases}$$

END OF NEXT TIME

PROP: FOR A MAP $f : X \to Y$, THE INDUCED MAPS $f^* : H^n(Y;R) \longrightarrow H^n(Y;R)$ SATISFY
$f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$

PROOF: ALREADY AT THE LEVEL OF COCHAINS:

$$f^\#(\varphi) \cup f^\#(\psi) = f^*(\varphi \cup \psi)$$

DIRECT CHECK: (RECALL: $f^\#(\varphi)(\sigma) = \varphi(f \circ \sigma)$)

$$f^\#\varphi \cup f^\#\psi = f^\#\varphi(\sigma|_{[v_0,\dots,v_k]})\, f^\#\psi(\sigma|_{[v_k,\dots,v_{k+l}]})$$
$$= \varphi(f \circ \sigma|_{\_\_\_})\,\psi(f \circ \sigma|_{\_\_\_\_})$$
$$= (\varphi \cup \psi)(f \circ \sigma) = f^\#(\varphi \cup \psi)(\sigma) \quad \square$$

NEXT TIME: COMPUTATIONS

ISSUE WITH DIRECT COMPUTATIONS: CAN'T REALLY DESCRIBE COCYCLES EXPLICITLY

TOOL: SIMPLICIAL (CO)HOMOLOGY

X Δ-CPLX. $\Delta_n(X)$ = FREE AB GP ON SPLICES OF Δ

SIMPLICIAL HOMOLOGY $H^\Delta_n(X) = H_n(\Delta_*(X))$.

SIMPL COHOM $H^n_\Delta(X) = H^*(\Delta^*(X))$

THM: INCLUSION $i : \Delta_n(X) \longrightarrow C_n(X)$ INDUCES ISO IN H_*

(IN AT1: ABSTRACT ISO, BUT WE SHOULD HAVE PROVED THIS))

COR: $i^* : H^n(X;R) \longrightarrow H^n_\Delta(X;R)$ IS ALSO ISO

PROOF OF COR: UCT + 5-LEMMA $\square$

NOTE: CUP MAKES SENSE IN SIMPL. COHOMOLOGY

UPSHOT: CAN COMPUTE CUP USING SIMPLICIAL COHOMOLOGY