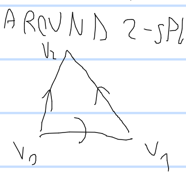
RECALL: CUP PRODUCT

- X IS Δ-CPLX IF $\exists\, \sigma_\alpha : \Delta^{n_\alpha} \to X$
  1) $\sigma_\alpha|_{\mathring{\Delta}_{n_\alpha}}$ INJ, $\forall x \in X\ \exists!\ \alpha$ $x \in \sigma_\alpha(\mathring{\Delta}_{n_\alpha})$
  2) $\sigma_\alpha|_{FACE} = \sigma_\beta$ ⟵ 1-SIMPLICES HAVE ARROWS, ARROUND 2-SIMPLICES:
  3) $A \subseteq X$ OPEN IFF $\sigma_\alpha^{-1}(A)$ OPEN $\forall \alpha$



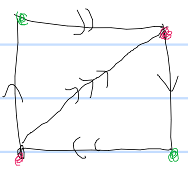
$\Delta_m(X)$ = FREE AB GP ON N-SIMPLICES

$\partial : \Delta_n(X) \to \Delta_{n-1}(X)$ $\quad \Sigma(-1)^i$ FACES

$H^m_\Delta(X;R) = H^m(\Delta_*(X);R)$

THM: WE CAN COMPUTE $\cup$ IN H^m_Δ

EXAMPLE 1: PROJECTIVE PLANE $\mathbb{RP}^2$,
$R = \mathbb{Z}/_2$ (FIRST EXAMPLE WITH $\mathbb{Z}/_2$ COEFFICIENTS SO WE DON'T HAVE TO WORRY ABOUT SIGNS)

$\mathbb{RP}^2 =$ 

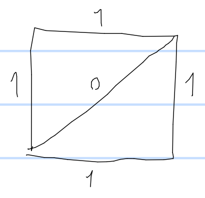
$$H^m(\mathbb{RP}^2, \mathbb{Z}/_2) = \begin{cases} \mathbb{Z}/_2 & m = 0,1,2 \\ 0 & \text{OTHERWISE} \end{cases}$$

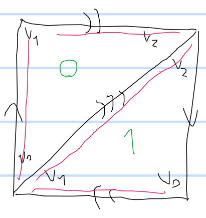
THE INTERESTING CASE IS
$H^1 \cup H^1 \to H^2$

E.G. $\cdot\ H^1 \cup H^2 \to H^3 = 0$
$\cdot\ H^0 = \{0, \text{IDENTITY OF CUP}\}$

NOTE: Δ^1 HAS 8 ELEMENTS

COCYCLE REPRESENTING THE NON-TRIVIAL CLASS $\alpha \in H^1$:



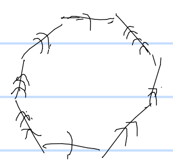
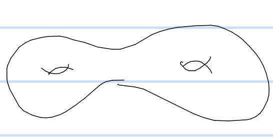
$\alpha \cup \alpha =$ 

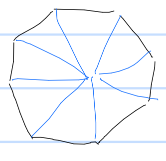
— = EVALUATE α

THIS REPRESENTS THE NON-TRIVIAL ELEMENT OF H^2
HENCE, UNDER THE ISO $H^1 \cong \mathbb{Z}/_2$, $H^2 \cong \mathbb{Z}/_2$,
$\cup : H^1 \times H^1 \to H^2$ IS $\mathbb{Z}/_2 \times \mathbb{Z}/_2 \to \mathbb{Z}/_2$
$\qquad a \quad b \mapsto ab$

LECTURE 5 NOTES

GENERALISATION TO HIGHER GENUS:

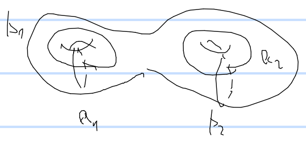
 = 

Δ-CPLX : 

$H^1 \cong \mathbb{Z}^{2g}$ GENERATED BY $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$

$H^2 \cong \mathbb{Z}$ GENERATED BY γ

- $\alpha_i \cup \beta_i = -(\beta_i \cup \alpha_i) = \gamma$
- ALL OTHER $\cup$ ARE 0.

REPRESENTATIVES OF GENERATORS OF H_1: 

α_i = "DUAL OF a_i"

FUNNY COINCIDENCE (NOT REALLY): $|\alpha_i \cup \beta_j|$ = # INTERSECTIONS OF a_i AND b_j.