RECALL: CUP PRODUCT

$\cup$ GIVES A "MULTIPLICATION IN COHOMOLOGY"
MORE PRECISELY:

LET $H^*(X;R) = \bigoplus H^n(X;R)$

THAT IS, ELEMENT ARE FINITE SUMS
$\sum_i \alpha_i$, $\alpha_i \in H^i(X;R)$

DEFINE: $\left(\sum_i \alpha_i\right)\left(\sum_j \beta_j\right) = \sum_{i,j} \alpha_i \cup \beta_j$ $(*)$

SHORTHAND NOTATION: $\alpha\beta = \alpha \cup \beta$

DIRECT CHECK: $H^*(X;R)$ WITH SUM AND $(*)$ IS A RING. IT IS A RING WITH 1 IF R IS A RING WITH 1.

EXERCISE: AS A RING, $H^*(\mathbb{RP}^2;\mathbb{Z}/_2)$ IS ISOMORPHIC TO $\mathbb{Z}/_2[\alpha]/(\alpha^3)$

QUOTIENT OF POLYNOMIAL RING $\mathbb{Z}/_2[\alpha]$ BY IDEAL GENERATED BY (α^3)

HINT: UNDER THE ISO, α IS THE NON-TRIVIAL ELEMENT OF $H^1(\mathbb{RP}^2;\mathbb{Z}/_2)$

$\alpha^2 = \alpha \cup \alpha =$ NON-TRIVIAL ELEM OF $H^2(\mathbb{RP}^2;\mathbb{Z}/_2)$

WE ALSO HAVE ADDITIONAL STRUCTURE:

DEFN: A <u>GRADED RING</u> IS A RING $A = \bigoplus_{k \geq 1} A_k$ SO THAT THE MULTIPLICATION TAKES $A_k \times A_l$ TO A_{k+l}

NOTATION: IF $a \in A_k$, WE WRITE $|a| = k$, AND CALL k THE <u>DIMENSION</u> OF a

NOTE
- DIMENSION IS NOT DEFINED FOR ALL $a \in A$
- SOMETIMES k AS ABOVE IS CALLED <u>DEGREE</u>

RMK: $H^*(X;R)$ IS A GRADED RING.

DEFN: A GRADED RING IS <u>COMMUTATIVE</u> IF

$ab = (-1)^{|a||b|}\, ba$ (WHENEVER $|a|, |b|$ ARE DEFINED)

NOTE: THIS TERMINOLOGY IS TERRIBLE.

IN THE CASE OF H^*:

THM: $H^*(X;R)$ IS A COMMUTATIVE GRADED RING IF R IS COMMUTATIVE
THAT IS, IF $\alpha \in H^k(X;R)$, $\beta \in H^l(X;R)$, THEN
$\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$

RMK: NOT TRUE FOR COCHAINS!

(1) $(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]})\,\psi(\sigma|_{[v_k,\dots,v_{k+l}]})$

(2) $(\psi \cup \varphi)(\sigma) = \psi(\sigma|_{[v_0,\dots,v_l]})\,\varphi(\sigma|_{[v_l,\dots,v_{k+l}]})$

— = NO CLEAR RELATION...

PROOF: LET $\varphi \in C^k(X;R)$, $\psi \in C^l(X;R)$
KEY OBSERVATION: (1) AND (2) DIFFER BY A PERMUTATION OF THE VERTICES OF Δ_{k+l}

LET $r_m: \Delta^m \to \Delta^m$ THE LINEAR HOMEO SO THAT $r_m(v_i) = v_{m-i}$

(r_m "REVERSES THE ORDER OF THE VERTICES")

LET $\varepsilon_m = (-1)^{m(m+1)/2}$

r_m IS COMPOSITION OF THIS MANY REFLECTIONS

FINALLY, LET $\rho: C_m(X) \to C_m(X)$ DEFINED BY $\rho(\sigma) = \varepsilon_m\, \sigma \circ r_m$

CLAIM: ρ IS A CHAIN-MAP, CHAIN-HOMOTOPIC TO id

ASSUME THE CLAIM FOR THE MOMENT:

CLAIM $\Rightarrow$ ρ INDUCES id IN H^*

COMPUTATIONS: $\varphi \in Z^k(X;R)$, $\psi \in Z^l(X;R)$

MORE PRECISELY, $\sigma|_{[v_0,\dots,v_k]} \circ r_k$

$(\rho^*\varphi \cup \rho^*\psi)(\sigma) = \varphi(\varepsilon_k\, \sigma|_{[v_k,\dots,v_0]})\ \psi(\varepsilon_l\, \sigma|_{[v_{k+l},\dots,v_k]})$

$\rho^*(\psi \cup \varphi)(\sigma) = \varepsilon_{k+l}\, \psi(\sigma|_{[v_{k+l},\dots,v_k]})\ \varphi(\sigma|_{[v_k,\dots,v_0]})$

R COMMUTATIVE $\Rightarrow$

$\varepsilon_k \varepsilon_l (\rho^*\varphi \cup \rho^*\psi) = \varepsilon_{k+l}\, \rho^*(\psi \cup \varphi)$

TAKE COHOM CLASSES: $\rho_* = id$

$\varepsilon_k \varepsilon_l ([\varphi] \cup [\psi]) = \varepsilon_k \varepsilon_l ([\rho^*\varphi] \cup [\rho^*\psi])$

$= \varepsilon_{k+l} [\rho^*(\psi \cup \varphi)] = \varepsilon_{k+l} [\psi \cup \varphi]$

NUMEROLOGY: $\varepsilon_{k+l} = (-1)^{kl} \varepsilon_k \varepsilon_l$, SO THIS IS THE REQUIRED EQUALITY

LEFT TO PROVE: CLAIM

ρ IS CHAIN-MAP (THAT IS, $\partial\rho = \rho\partial$):

COMPUTATION:

$\partial(\rho(\sigma)) = \varepsilon_m \sum (-1)^i \sigma|_{[v_m,\dots,\hat{v}_{m-i},\dots v_0]}$

$\rho(\partial\sigma) = \rho\left(\sum (-1)^i \dots\right)$

$= \varepsilon_{m-1} \sum (-1)^i \sigma|_{[v_m,\dots,\hat{v}_i,\dots v_0]}$

CHANGE INDICISATION IN SUM

$= \varepsilon_{m-1} \sum (-1)^{m-i} \sigma|_{[v_m,\dots,\hat{v}_{m-i},\dots v_0]}$

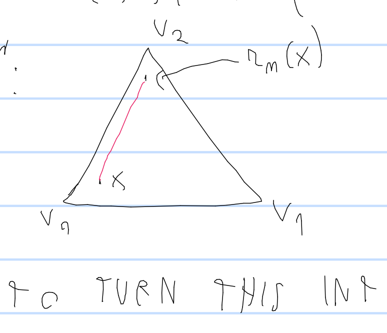
$\varepsilon_m = \varepsilon_{m-1}(-1)^m \Rightarrow$ OK ✓

NOTE: WE REALLY NEEDED ε_m TO MAKE ρ A CHAIN-MAP.

ρ IS CHAIN HOMOTOPIC TO id:

HEURISTIC PICTURE

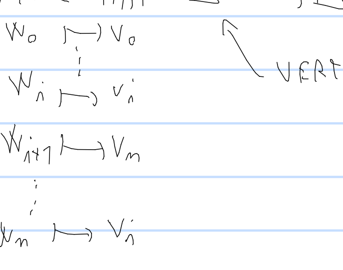
$P(\sigma) =$



CAN BE FILLED IN, IN A NATURAL WAY, AND SUBDIVIDED

NATURAL WAY: $(x,t) \mapsto \sigma((1-t)x + t\,r_m(x))$

"FROM TOP":



IF ONE TRIES TO TURN THIS INTO FORMULAS, ONE COMES UP WITH:

NOTATION: D_i = AFFINE MAP $\Delta^{m+1} \to \Delta^m$ WITH

VERTICES = w_i VERTICES = v_i

$w_0 \mapsto v_0$
$\vdots$
$w_i \mapsto v_i$
$w_{i+1} \mapsto v_m$
$\vdots$
$w_m \mapsto v_i$

LET $P(\sigma) = \sum_i (-1)^i \varepsilon_{m-i}\, \sigma \circ D_i$

LET'S CHECK THAT P IS CHAIN HOMOTOPY

NOTATION: $\sigma \circ D_i = [v_0,\dots,v_i,v_m,\dots v_i]$

$\partial P = \sum_j \sum_i \dots =$

$= \sum_{j \leq i}^{(*)} (-1)^i (-1)^j \varepsilon_{m-i} [v_0,\dots,\hat{v}_j,\dots v_i, v_m,\dots,v_i]$

$\overset{(**)}{+} \sum_{j \geq i} (-1)^i (-1)^{i+1+m-j} \varepsilon_{m-i} [v_0,\dots,v_i,v_m,\dots\hat{v}_j,\dots v_i]$

THE TERMS $j = i$ GIVE:

$\underbrace{\varepsilon_m [v_m,\dots,v_0]}_{i=0 \text{ IN } (*)} + \sum_{i>0} \varepsilon_{m-i} [v_0,\dots,v_{i-1},v_m,\dots,v_i]$

$+ \sum_{i<m} (-1)^{m+i-1} \varepsilon_{m-i} [v_0,\dots,v_i,v_m,\dots,v_{i+1}]$

$\underbrace{-[v_0,\dots,v_m]}_{i=m \text{ IN } (**)}$

THE MIDDLE TERMS CANCEL OUT (SINCE $(-1)^{m+i} \varepsilon_{m-i+1} = -\varepsilon_{m-i}$)

THE OTHER TERMS ARE $\rho(\sigma) - \sigma$.

FINAL CHECK: THE TERMS $j \neq i$ GIVE $-P\partial$

THIS IS ANOTHER DIRECT COMPUTATION USING DEFINITIONS $\square$