RECALL: COMMUTATIVE GRADED RINGS

EXAMPLES

1) <u>POLYNOMIAL RINGS</u> $R[\alpha]$ AND "TRUNCATED VERSIONS" THEREOF $R[\alpha]/(\alpha^n)$

NOTE: ELEMENTS OF $R[\alpha]/(\alpha^n)$ ARE (REPRESENTED BY ELEMENTS OF $R[\alpha]$) OF THE FORM $c_0 + c_1\alpha + c_2\alpha^2 + \dots + c_{n-1}\alpha^{n-1}$

EXERCISE: $H^*(\mathbb{RP}^2; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^3)$, WITH $|\alpha| = 1$

WE WILL SHOW: $\cdot\, H^*(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$, $|\alpha| = 1$

$H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$, WITH $|\alpha| = 2$

WE (PROBABLY) WON'T SHOW: $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) \cong \mathbb{Z}_2[\alpha]$, WITH $|\alpha| = 1$

$\cdot\, H^*(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$, $|\alpha| = 2$

DIGRESSION: CUP PRODUCTS DETECT MORE

- $S^2 \vee S^4$ AND $\mathbb{CP}^2$ HAVE ISOMORPHIC π_1, H_n, H^n $\forall n$
- HOWEVER: $H^*(\mathbb{CP}^2, \mathbb{Z})$ HAS α WITH $|\alpha| = 2$, AND $S^2 \vee S^4$ DOES NOT. $\Rightarrow S^2 \vee S^4$ AND $\mathbb{CP}^2$ ARE NOT HOMOTOPY EQUIVALENT

SIMILAR EXAMPLE: $S^2 \times S^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$

2) <u>EXTERIOR ALGEBRAS</u> $\Lambda_R[\alpha_1, \dots, \alpha_n]$

NOTATION: FOR $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, n\}$, LET

$$\alpha^I = \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_k}$$

ELEMENTS OF $\Lambda_R[\alpha_1, \dots, \alpha_n] = \sum c_I \alpha^I$

MULTIPLICATION DEFINED BY RULES. $\alpha_i\alpha_j = -\alpha_j\alpha_i \quad i \neq j$

$\cdot\, \alpha_i^2 = 0$

$\cdot$ ASSOCIATIVE + DISTRIBUTIVE

E.G.: $(\alpha_1\alpha_3 + \alpha_2) \cdot \alpha_2 = -\alpha_1\alpha_2\alpha_3 \, (+0)$

TO MAKE IT GRADED: $|\alpha_i|$ ODD

EXERCISE: $H^*(\underset{S^1 \times S^1}{\underset{\shortparallel}{T^2}}; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha, \beta]$, WITH $|\alpha| = |\beta| = 1$

WE'LL SEE: $H^*(T^n; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n]$ WITH $|\alpha_i| = 1$

$\cdot$ (MORE GENERAL)

$H^*(\text{PRODUCT OF SPHERES}; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n]$

$|\alpha_i|$ = DIMENSION OF i-TH SPHERE

3) PRODUCT RINGS. THERE IS RING ISO

$$H^*(X \sqcup Y; R) \cong H^*(X; R) \oplus H^*(Y; R)$$

TO COMPUTE MORE EXAMPLES WE NEED TO UNDERSTAND: <u>COHOMOLOGY OF PRODUCTS.</u>

DEFN: THE <u>CROSS PRODUCT</u>

$$H^*(X; R) \times H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$$

IS GIVEN BY $a \times b = p_X^*(a) \cup p_Y^*(b)$ $\quad p_X, p_Y$ PROJECTIONS ONTO FACTORS.

RMK: $\times$ IS BILINEAR $\quad (a+b) \times c = a \times c + b \times c$

$a \times (b+c) = a \times b + a \times c$

$\times$ BILINEAR $\Rightarrow$ $\times$ IS VERY RARELY HOMOMORP M, EVEN LESS OFTEN ISO

FIX: TENSOR PRODUCT.

RECALL: $\cdot$ A, B AB GPS

$\cdot\, A \otimes B$ = AB GP GENERATED BY $\{a \otimes b : a \in A, b \in B\}$ AND RELATIONS. $(a+a') \otimes b = a \otimes b + a' \otimes b$

$\cdot\, a \otimes (b+b') = a \otimes b + a \otimes b'$

$\cdot$ A, B R-MODULE, R COMMUTATIVE $\Rightarrow$ $A \otimes_R B$ HAS THE FURTHER RELATIONS $ra \otimes b = a \otimes rb$

LEMMA/EXERCISE: A BILINEAR MAP $\varphi : A \times B \to C$ INDUCES A HOMO $A \otimes B \to C$ SENDING $a \otimes b$ TO $\varphi(a, b)$

LEMMA $\Rightarrow$ $\times$ AS DEFINED ABOVE GIVES A HOMO

$$H^*(X; R) \otimes_R H^*(Y; R) \longrightarrow H^*(X \times Y; R)$$
$$a \otimes b \longmapsto a \times b$$

THIS MAP IS ALSO DENOTED $\times$, AND CALLED CROSS PRODUCT.

LEMMA: A, B GRADED RINGS, AND R-MODULES $\Rightarrow$ $A \otimes_R B$ HAS THE STRUCTURE OF A RING WITH MULTIPLICATION

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$$

ALSO, IT IS A GRADED RING:

$$A \otimes_R B = \bigoplus_k \Big(\bigoplus_{i+j=k} A_i \otimes_R B_j \Big)$$

REASON TO DEFINE MULTIPLICATION LIKE THIS: $\times$ BECOMES A RING HOMO. INDEED:

$$\times\big((a \otimes b)\,(c \otimes d)\big) \;\Big(= \times\big((-1)^{|b||c|} (a \cup c) \otimes (b \cup d)\big) = (-1)^{|b||c|} p_X^*(a \cup c) \cup p_Y^*(b \cup d) = \Big) \text{ SKIP}$$

$$= (-1)^{|b||c|} p_X^*(a) \cup p_X^*(c) \cup p_Y^*(b) \cup p_Y^*(d)$$

$$= p_X^*(a) \cup p_Y^*(b) \cup p_X^*(c) \cup p_Y^*(d)$$

$$= \times(a \otimes b) \times (c \otimes d)$$

THM: $H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$ IS A RING ISO IF X, Y ARE CW-CPLX, AND $H^k(Y; R)$ IS FIN GEN, R-MODULE $\forall k$

FREE