RECALL:

CROSS PRODUCT

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R) \quad (1)$$

$$\alpha \otimes \beta \longmapsto p_X^*(\alpha) \cup p_Y^*(\beta)$$

$p_X : X \times Y \to X$

$p_Y : X \times Y \to Y$

THM (KÜNNETH FORMULA) (1) IS AN ISOMORPHISM OF GRADED RINGS IF
- X, Y ARE CW-CPLXS
- $H^k(Y;R)$ IS FIN GEN FREE R-MODULE $\forall k$

EXAMPLE OF TENSOR PRODUCT (+ CORRECTION TO PREVIOUS LECTURE)

$\Lambda_R[\alpha_1, \alpha_2] \cong \Lambda_R[\alpha_1] \otimes \Lambda_R[\alpha_2]$, IF THE α_i HAVE ODD DEGREE.

KÜNNETH FORMULA $\Rightarrow$

$$H^*(S^{k_1} \times \dots S^{k_n}; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_n]$$

IF ALL k_i ARE ODD

TO INCLUDE EVEN-DIM SPHERES: SAY k_i ODD $\Leftrightarrow i \leq j$

$$\Rightarrow H^*(S^{k_1} \times \dots S^{k_n}; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[\alpha_1, \dots, \alpha_j] \otimes_R \mathbb{Z}[\alpha_{j+1}]/(\alpha_{j+1}^2) \dots \otimes_R \mathbb{Z}[\alpha_n]/(\alpha_n^2)$$

NOTE: AS RINGS, $\Lambda_{\mathbb{Z}}[\alpha] \cong \mathbb{Z}[\alpha_m]/(\alpha_m^2)$

EXERCISE: PROVE ALL THIS.

TODAY (AND FRIDAY?): OUTLINE/HIGHLIGHTS OF PROOF OF THM

TOOL WE NEED: $\cup$ IN RELATIVE H^*

IN SHORT: EVERYTHING WORKS THE SAME

RECALL:

$$\varphi \cup \psi(\sigma) = \varphi(\sigma|_{[\dots]})\, \psi(\sigma|_{[\dots]})$$

IF φ OR ψ VANISHES ON SING SIMPLICES IN A, THEN SO DOES $\varphi \cup \psi \Rightarrow$

$\cup : C^k \times C^l \to C^{k+l}$ INDUCES CUP PRODUCTS:

$$H^k(X, A; R) \times H^l(X, A; R) \longrightarrow H^{k+l}(X, A; R)$$

= INCLUDE ONE OR BOTH

STILL TRUE:
- $H^*(X, A; R)$ IS GRADED RING
- " IS COMMUT. GRADED IF R IS COMMUTATIVE
- $\exists$ CROSS PRODUCT, SAME DEFN

STRATEGY TO PROVE THM:

GRADING OF $H^*(X,A;R) \otimes_R H^*(Y;R)$

DEFINE
- $h^n(X, A) := \bigoplus_i \left(H^i(X, A; R) \otimes_R H^{n-i}(Y; R)\right)$
- $k^n(X, A) := H^n(X \times Y, A \times Y; R)$
- $\mu : h^n(X, A) \to k^n(X, A)$ GIVEN BY CROSS PROD

WOULD LIKE TO SHOW:

LEMMA 0: $\mu : h^n(\{pt\}, \emptyset) \to k^n(\{pt\}, \emptyset)$ IS ISO $\forall n$

(LEMMA 0 IS EASY)

LEMMA 1: h^*, k^* ARE UNREDUCED COHOM. THEORIES, THAT IS, ALL AXIOMS HOLD EXCEPT DIMENSION (TO BE PRECISE, NOTE THAT $\exists$ INDUCED MAPS + CONNECTING HOMO)

LEMMA 2: μ IS NATURAL TRANSFORMATION:

$$\begin{array}{ccc} h^n(X,A) & \longrightarrow & h^n(Y,B) \\ \mu \downarrow & & \downarrow \mu \\ k^n(X,A) & \longrightarrow & k^n(Y,B) \end{array}$$

COMMUTES + SIMILARLY FOR CONNECTING HOMO

WE WANT THIS BECAUSE:

PROP: A NATURAL TRANSF. BETWEEN UNRED. COHOM. THEORIES WHICH IS AN ISO FOR $(\{pt\}, \emptyset)$ IS AN ISO $\forall$ CW PAIRS.

SO: LEMMAS + PROP $\Rightarrow$ THM

HIGHLIGHTS OF PROOFS:

LEMMA 1:

ADDITIVITY AXIOM FOR h^*:

ALGEBRAIC FACT: M_α R-MODULES, N FIN GEN FREE R-MODULE

THEN $\left(\prod_\alpha M_\alpha\right) \otimes_R N \cong \prod_\alpha (M_\alpha \otimes_R N)$

INDEED, $M_\alpha \otimes_R N$ IS DIRECT PRODUCT OF FIN. MANY $M_{\alpha\beta} \cong M_\alpha$ $\quad (M \otimes_R R^n \cong M^n)$

$$\left(\prod_\alpha M_\alpha\right) \otimes_R N \cong \prod_\beta \prod_\alpha M_\alpha \cong \prod_\alpha \prod_\beta M_{\alpha\beta} \cong \prod_\alpha (M_\alpha \otimes_R N)$$

(FREENESS IS ALSO USED FOR EXACTNESS)

SKETCH PROOF OF PROP, FOR X FIN DIM:

RECALL:
- EXACTNESS AXIOM

$$\dots \to h^i(X,A) \to h^i(X) \to h^i(A) \to h^{i+1}(X,A) \to \dots$$

- 5 LEMMA:

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

MIDDLE MAP IS ISO IF THE OTHER MAPS ARE.

5-LEMMA $\Rightarrow$ ENOUGH TO DO THE CASE $A = \emptyset$

DIM = 0: HYPOTHESIS + ADDITIVITY AXIOM.

INDUCTION: EXACTNESS + 5-LEMMA GIVE:

$\mu : h^*(X^{(n)}) \to k^*(X^{(n)})$ ARE ISO

PROVIDED THAT $\mu : h^*(X^{(n)}, X^{(n-1)}) \to k^*(X^{(n)}, X^{(n-1)})$

ARE ISO

$$h^i(X^{(n)}, X^{(n-1)}) \cong \prod h^i(D^n_\alpha, \partial D^n_\alpha)$$
$$k^i(X^{(n)}, X^{(n-1)}) \cong \text{— — — — —}$$

(NATURAL ISOS)

$h^i(D^n, \partial D^n) \cong k^i(D^n, \partial D^n)$ BECAUSE:
- $h^i(D^n) \cong k^i(D^n)$ SINCE D^n IS CONTRACTIBLE
- EXACTNESS + 5-LEMMA $\square$

THERE IS ALSO REL VERSION OF KÜNNETH FORM:

THM $H^*(X, A; R) \otimes_R H^*(Y, B; R) \to H^*(X \times Y; A \times Y \cup X \times B; R)$

GIVEN BY CROSS PRODUCT IS ISO IF $(X, A), (Y, B)$ ARE CW-PAIRS AND $H^k(Y, B; R)$ IS FIN GEN FREE R-MODULE $\forall k$

(USES: MORE GENERAL RELATIVE CUP PRODUCT)

NEXT LECTURE: H^* (PROJECTIVE SPACES)