Exercise Sheet 2

1. The von Mangoldt function $\Lambda : \mathbb{N} \to \mathbb{R}$ is defined by

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^\nu \text{ is a prime power } (\nu \geq 1), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover we define, for $x \geq 1$,

$$\psi(x) := \sum_{n \leq x} \Lambda(n).$$

In this exercise, we will assume the following strong form of the Prime Number Theorem: [1]

$$\psi(x) = x + O\left(\frac{x}{(\log x)^2}\right) \qquad (x \geq 2).$$

a.) Show that for any $n \in \mathbb{N}$ we have

$$\sum_{d \mid n} \Lambda(d) = \log n.$$

b.) Using the summation by parts formula (Lemma A.1.1), prove that for $x \geq 1$ we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \int_1^x \frac{\psi(t)}{t^2}\, dt + \frac{\psi(x)}{x}.$$

c.) Using the above version of the Prime Number Theorem, show that uniformly over $x \geq 2$, we have

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + A + O\left(\frac{1}{\log x}\right),$$

where

$$A = \int_1^\infty \frac{\psi(t) - t}{t^2}\, dt + 1$$

and where the integral converges absolutely.

[1] One can show that this follows e.g. from

$$\pi(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + O\left(\frac{x}{(\log x)^3}\right).$$

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d.) Prove that, uniformly over $x \geq 3$, we have

$$\sum_{n \leq x} \frac{\Lambda(n) \log n}{n} = \frac{1}{2}(\log x)^2 + O(\log \log x).$$

e.) You might have seen before that, uniformly over $x \geq 1$, we have

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right),$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant. Use this in combination with a.), c.) and d.) to show that

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 + (A + \gamma) \log x + O(\log \log x).$$

f.) Show that, on the other hand, we have

$$\sum_{n \leq x} \frac{\log n}{n} = \frac{1}{2}(\log x)^2 + O(1)$$

and conclude that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x - \gamma + O\left(\frac{1}{\log x}\right)$$

as well as

$$\int_1^\infty \frac{t - \psi(t)}{t^2}\, dt = 1 + \gamma.$$

2. Solve Exercise 1.4.4 from the lecture notes.