Solution to Exercise Sheet 1

Exercise 1.

a.) The claim obviously holds for $n = 1$. By playing around with special cases, one can already see some patterns: For example,

$$\sum_{d \mid 30} \mu(d) = \mu(1)+(\mu(2) + \mu(3) + \mu(5))+(\mu(6) + \mu(10) + \mu(15))+\mu(30) = 1-3+3-1 = 0.$$

Indeed, if n is squarefree, $n = p_1 \cdots p_l$ with distinct primes p_l and $l \geq 1$, then

$$\sum_{d \mid n} \mu(d) = \sum_{I \subseteq \{1,\dots,l\}} (-1)^{|I|} = \sum_{i=1}^{l} (-1)^i \binom{l}{i} = (1-1)^l = 0.$$

Now if $n = p_1^{e_1} \cdots p_l^{e_l}$ with distinct primes p_i and $e_i \in \mathbb{N}$, denote $r(n) = p_1 \cdots p_l$. Then $d \mid n$ and $\mu(d) \neq 0$ if and only if $d \mid r(n)$, so

$$\sum_{d \mid n} \mu(d) = \sum_{d \mid r(n)} \mu(d) = 0,$$

as claimed.

b.) Using a.) and the fact that $d \mid (k, n)$ if and only if $d \mid n$ and $d \mid k$, we see that

$$\varphi(n) = \sum_{\substack{1 \leq k \leq n \\ (k,n)=1}} 1 = \sum_{1 \leq k \leq n} \sum_{d \mid (k,n)} \mu(d) = \sum_{d \mid n} \mu(d) \sum_{\substack{1 \leq k \leq n \\ k \equiv 0\, (d)}} 1 = n \sum_{d \mid n} \frac{\mu(d)}{d}.$$

□

Exercise 2.

a.) Note that

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right) = \sum_{kl=n} f(k) g(l) = (g * f)(n).$$

Moreover, we have

$$
\begin{aligned}
((f*g)*h)(n) &= \sum_{kl=n}(f*g)(k)h(l) = \sum_{kl=n}\sum_{ab=k} f(a)g(b)h(l) = \sum_{abc=n} f(a)g(b)h(c) \\
&= \sum_{kl=n}\sum_{bc=l} f(k)g(b)h(c) = (f*(g*h))(n).
\end{aligned}
$$

b.) Let

$$
1(n) = 1 \text{ for all } n \qquad \text{and} \qquad \delta(n) := \begin{cases} 1, & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}
$$

then 1a.) tells us exactly that $\mu * 1 = \delta$. Moreover, it is clear that for any f we have $f * \delta = f$. Hence,

$$
\begin{aligned}
g(n) = \sum_{d\,|\,n} f(d) \text{ for all } n \Leftrightarrow g = f * 1 \Rightarrow g * \mu = f * 1 * \mu = f * \delta = f \\
\Leftrightarrow f(n) = \sum_{d\,|\,n} g(d)\mu\left(\frac{n}{d}\right) \text{ for all } n
\end{aligned}
$$

and conversely

$$
f = g * \mu \Rightarrow f * 1 = g * \mu * 1 = g * \delta = g.
$$

Note that we did not need to use parentheses for triple (or longer) Dirichlet convolutions due to associativity.

c.) Denoting by Id, the identity on $\mathbb{N}$, exercise 1b.) tells us that $\varphi = \mathrm{Id} * \mu$. By Möbius inversion, this implies

$$
n = \mathrm{Id}(n) = (\varphi * 1)(n) = \sum_{d\,|\,n} \varphi(d)
$$

as claimed.

d.) We have

$$
\begin{aligned}
F(s)G(s) &= \left(\sum_{n\geq 1}\frac{f(n)}{n^s}\right)\left(\sum_{m\geq 1}\frac{g(m)}{m^s}\right) = \sum_{n,m\geq 1}\frac{f(n)g(m)}{(nm)^s} \\
&= \sum_{k\geq 1}\frac{1}{k^s}\sum_{nm=k} f(n)g(m) = \sum_{k\geq 1}\frac{(f*g)(k)}{k^s}.
\end{aligned}
$$

□

Siehe nächstes Blatt!

Exercise 3.

a.) For $\Re s > 1$, we have

$$\sum_{n\geq 1} \frac{\mu(n)}{n^s} \leq \sum_{n\geq 1} \frac{1}{n^{\Re s}} < \infty,$$

so both Dirichlet series converge absolutely in the required range. Therefore, we obtain

$$1 = \sum_{n\geq 1} \frac{\delta(n)}{n^s} = \sum_{n\geq 1} \frac{(1*\mu)(n)}{n^s} = \zeta(s) \sum_{n\geq 1} \frac{\mu(n)}{n^s}$$

as claimed.

b.)

$$\begin{aligned}\mathbb{P}[(X_N, Y_N) = 1] &= \frac{1}{N^2} \sum_{n_1,n_2\leq N} \mathbb{1}((n_1,n_2)=1) = \frac{1}{N^2} \sum_{n_1,n_2\leq N} \sum_{d\,|\,(n_1,n_2)} \mu(d) \\ &= \frac{1}{N^2} \sum_{d\leq n} \mu(d) \sum_{\substack{n_1\leq N \\ n_1\equiv 0\,(d)}} \sum_{\substack{n_2\leq N \\ n_2\equiv 0\,(d)}} 1 = \frac{1}{N^2} \sum_{d\leq N} \mu(d) \left[\frac{N}{d}\right]^2.\end{aligned}$$

But $\left[\frac{N}{d}\right] = \frac{N}{d} + O(1)$, hence the last term is

$$= \sum_{d\leq N} \frac{\mu(d)}{d^2} + O\left(\frac{1}{N} \sum_{d\leq N} \frac{1}{d}\right) = \sum_{d\leq N} \frac{\mu(d)}{d^2} + O\left(\frac{\log N}{N}\right).$$

Moreover, we have

$$\sum_{d\leq N} \frac{\mu(d)}{d^2} = \sum_{d\geq 1} \frac{\mu(d)}{d^2} + O\left(\sum_{d>N} \frac{1}{d^2}\right) = \frac{1}{\zeta(2)} + O\left(\frac{1}{N}\right)$$

using a.). We thus conclude that $\mathbb{P}[(X_N, Y_N) = 1] \to \frac{1}{\zeta(2)}$ as $N \to \infty$, as claimed.

c.) We have

$$\sum_{n\geq 1} \frac{1}{(2n-1)^2} + \frac{\zeta(2)}{4} = \sum_{n\geq 1} \frac{1}{(2n-1)^2} + \sum_{n\geq 1} \frac{1}{(2n)^2} = \sum_{n\geq 1} \frac{1}{n^2} = \zeta(2)$$

and therefore

$$\zeta(2) = \frac{4}{3} \sum_{n\geq 1} \frac{1}{(2n-1)^2}.$$

Bitte wenden!

d.) Note that, using non-negativity (Tonelli) to exchange the orders of integration and summation,

$$I = \int_0^1 \int_0^1 \frac{1}{1 - x^2y^2}\, dx\, dy = \int_0^1 \int_0^1 \sum_{n\geq 1} (x^2y^2)^{n-1}\, dx\, dy = \sum_{n\geq 1} \left(\int_0^1 x^{2n-2}\, dx \right)^2 = \sum_{n\geq 1} \frac{1}{(2n-1)^2}.$$

e.) The Jacobi determinant of the coordinate transformation is given by

$$\det \begin{pmatrix} \frac{\cos u}{\cos v} & \frac{\sin u \sin v}{\cos^2 v} \\ \frac{\sin u \sin v}{\cos^2 u} & \frac{\cos v}{\cos u} \end{pmatrix} = 1 - x^2y^2.$$

Moreover, one verifies that the range of integration transforms to $0 < v < \frac{\pi}{2}$ and $0 < u < \frac{\pi}{2} - v$ (a right triangle with side lengths $\frac{\pi}{2}$ adjacent to the right angle), hence

$$I = \int_0^{\pi/2} \int_0^{\pi/2 - v} 1\, du\, dv = \frac{\pi^2}{8}.$$

□

We remark that this argument can in fact be generalised to give the value of $\zeta(2k)$ for any positive integer k. The interested reader is invited to check [Elk03] for the details.

Literatur

[Elk03] Noam D. Elkies. On the sums $\sum_{k=-\infty}^{\infty}(4k+1)^{-n}$. *Amer. Math. Monthly*, 110(7):561–573, 2003.