Schmidt's game applied to the set of badly approximable numbers

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Notation 1. *For any set $X \subset \mathbb{R}^n$ we denote with $\rho(X)$ its diameter, i.e. $\rho(X) = max\{d(x,y)|x,y \in X\}$.*

Ayesha (A) and Bhupen (B) are playing a game together. The game has the following rules:

- They fix $m \in \mathbb{N}_{>0}$.
- A chooses a number $0 < \alpha < \frac{1}{2}$ and B chooses a number $0 < \beta < 1$.
- B chooses a closed ball $B_0 \subset \mathbb{R}^m$.
- A chooses a closed ball $A_0 \subset B_0$ such that $\rho(A_0) = \alpha\rho(B_0)$
- B chooses a closed ball $B_1 \subset A_0$ such that $\rho(B_1) = \beta\rho(A_0)$
- Now A and B play alternatively: If it is A's turn, she chooses a closed ball $A_n \subset B_n$ such that $\rho(A_n) = \alpha\rho(B_n)$ If it is B's turn, he chooses a closed ball $B_n \subset A_{n-1}$ such that $\rho(B_n) = \beta\rho(A_{n-1})$

This game is called Schmidt's game after its inventor.

Definition 2. *A set $S \subset \mathbb{R}^m$ is called (α,β)-winning if A can play in such a way that the unique intersection point $\cap_{n=0}^{\infty} A_n = \cap_{n=0}^{\infty} B_n \in S$ regardless of how B is playing. A set is called α-winning if it is (α,β)-winning for all $\beta \in (0,1)$ and it is called winning if it is α-winning for some α.*

Informally B tries to stay away from the target S whilst A tries to land on it.

Example 3. *Trivially $S = \mathbb{R}^m$ is winning.*

In the first theorem, we want to show a more interesting example of a winning set.

Before doing so, we want to state two properties of winning sets which we will not proof:

Theorem 4. *The Hausdorff dimension of a winning set is maximal.*

Theorem 5. *The image by a C^1-diffeomorphism of a winning set is again a winning st.*

Theorem 6 (Schmidt, 1966)**.** *The set* $S := \{x \in \mathbb{R}$ *such that* $\exists c(x) > 0 : \|qx\| > \frac{c(x)}{q} \forall q \in \mathbb{N}\}$ *(i.e. the set of badly approximable numbers) is winning.*

Proof. We can restrict ourselves to the unit interval $[0,1]$. Indeed since $\alpha, \beta < 1$, there exists $n, a, l \in \mathbb{Z}$ such that $\rho(B_n) < 1$ and $B_n \subset [a + \frac{1}{l}, a + 1 + \frac{1}{l}]$. We saw that an irrational number is badly approximable if and only if the coefficients of its continued fraction are bounded. Thus we can construct a bijection between the badly approximable numbers in the unit interval and $[a + \frac{1}{l}, a + 1 + \frac{1}{l}]$.

Since $0 < \alpha < \frac{1}{2}$ and $0 < \beta < 1$, we have that $\alpha\beta < \frac{1}{2}$. Thus

$$R := (\alpha\beta)^{-1} > 2$$

For all $n \in \mathbb{Z}_{\geq 0}$ we define

$$Q_n = \{\frac{p}{q} \text{ where } gcd(p,q) = 1 \quad : R^{\frac{n-3}{2}} \leq q < R^{\frac{n-2}{2}}\} \subset \mathbb{Q}$$

Note that if $n \leq 2$ then $\frac{n-2}{2} \leq 0$, thus $R^{\frac{n-2}{2}} \leq 1$ and therefore $Q_0 = Q_1 = Q_2 = \emptyset$. The sets Q_n are disjoint and $\mathbb{Q} = \cup_{n=3}^{\infty} Q_n$.

Let $\frac{p}{q} \neq \frac{p'}{q'} \in Q_n$. Note that $|pq' - p'q|$ has to be a nonzero integer, thus $|pq' - p'q| \geq 1$. Hence

$$|\frac{p}{q} - \frac{p'}{q'}| = |\frac{pq' - p'q}{qq'}| \geq \frac{1}{|qq'|} > \frac{1}{(R^{\frac{n-2}{2}})^2} = R^{-n+2} \tag{1}$$

For $\frac{p}{q} \in \mathbb{Q}_n$ we define the dangerous interval as follows:

$$\Delta(\frac{p}{q}) := \{x \in [0,1] : |x - \frac{p}{q}| < \delta R^{-n}\}$$

where $0 < \delta < 1$ depends on α and on the first move of B and will be specified later.

Note that the dangerous interval of a fraction is either "in the middle of" the unit interval or at its boarder. (See figure 1.)

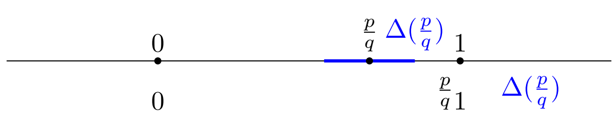

Figure 1: Two possibilities for the dangerous interval

Our goal is to show that there exists a strategy for A such that for all $n \in \mathbb{N}$ and $\forall \frac{p}{q} \in Q_n$, we have

$$A_n \cap \Delta(\frac{p}{q}) = \emptyset \tag{2}$$

.

This implies that $x \in \cap_{n=0}^{\infty} A_n$ satisfies for all $p, q \in \mathbb{Z}$ that $|x - \frac{p}{q}| \geq \delta R^{-n}$. Thus x is badly approximable, i.e. $x \in Bad(1)$.

Let $B_0 \subset [0,1]$ be any closed ball. Set $\delta := \rho(B_0)(\frac{1}{2} - \alpha)$

We define A_n inductively. Thus assume that for all $j < n$ A_j is a closed interval such that equation (2) holds. If $B_n \cap \Delta(\frac{p}{q}) = \emptyset$, then A can choose any closed interval of B_n with diameter $\alpha\rho(B_n)$ Otherwise we claim that there exists a unique point $\frac{p_n}{q_n} \in Q_n$ such that $\Delta(\frac{p_n}{q_n})$ intersects B_n. Indeed assume there are $\frac{p}{q} \neq \frac{p'}{q'} \in Q_n$ and x, x' such that $x \in \Delta(\frac{p}{q}) \cap B_n$ and $x' \in \Delta(\frac{p'}{q'}) \cap B_n$. Then

$$\begin{aligned} |\frac{p}{q} - \frac{p'}{q'}| &= |\frac{p}{q} - x + x - x' + x' - \frac{p'}{q'}| \\ &\overset{(a)}{\leq} |\frac{p}{q} - x| + |x - x'| + |x' - \frac{p'}{q'}| \overset{(b)}{\leq} 2\delta R^{-n} + 1 \\ &\overset{(c)}{=} 2\rho(B_0)(\frac{1}{2} - \alpha)R^{-n} + 1 \overset{(d)}{\leq} R^{-n} + 1 \overset{(e)}{\leq} R^{-n+2} \end{aligned} \tag{3}$$

where (a) uses the triangle inequality, (b) follows since $x, x' \in \Delta(\frac{p}{q})$ and $x, x' \in B_n \subset [0,1]$, (c) uses the definition of δ, (d) follows since $\rho(B_0) \leq 1$ and $0 < \alpha < \frac{1}{2}$ and (e) since $R > 2$. But $|\frac{p}{q} - \frac{p'}{q'}| \leq R^{-n+2}$ contradicts equation (1) and thus proves the claim.

Therefore it suffices to consider two cases

1. The dangerous $\Delta(\frac{p_n}{q_n})$ divides B_n in two intervals as in the first diagram of figure 1.

   Note that the diameter of B_n is $(\alpha\beta)^n \rho(B_0) = R^{-n}\rho(B_0)$. The larger of both intervals has length bigger or equal than

$$\begin{aligned} \frac{1}{2}(\rho(B_n) - \rho(\Delta(\frac{p_n}{q_n}))) &= \frac{1}{2}(\rho(B_n) - 2\delta R^{-n}) \\ = \frac{1}{2}(\rho(B_n) - 2R^{-n}\rho(B_0)(\frac{1}{2} - \alpha)) &= \frac{1}{2}(\rho(B_n) - (1 - 2\alpha)\rho(B_n)) \\ &= \alpha\rho(B_n) \end{aligned}$$

2. The dangerous $\Delta(\frac{p_n}{q_n})$ does not divide $[0,1]$ as in the second diagram of figure 1.

   Then it is trivially possible to choose a closed interval $A_n \subset B_n \setminus \Delta(\frac{p_n}{q_n})$ of length $\alpha\rho(B_0)$.

Thus we found a strategy such that condition (2) holds. □

Note that we proved a stronger statement than the theorem: We did not show that there is some choice of α such that the set $Bad(1)$ is winning, we showed that it is winning for any choice of α.

Theorem 4 Together with theorem 6 imply Jarnik's theorem.

In the second theorem we want to observe how the intersection of winning sets behave.

Theorem 7. *The intersection of countably many α-winning sets is α-winning.*

Before we can to prove this theorem, we need to formalize the notation of a winning strategy.

Let S be an α-winning set. Fix $0 < \beta < 1$. Then we can find functions $f_0, f_1, f_2, f_3, ...$ such that for $i \in \mathbb{N}$, f_i takes as arguments closed balls $B_0, B_1, ..., B_i$ where $\rho(B_j) = (\alpha\beta)^j \rho(B_0)$ and $B_0 \supset B_1 \supset ... \supset B_i$. The image of $f_i(B_0, ..., B_i)$ is a closed ball of diameter $\alpha(\alpha\beta)^i$ contained in B_i. Moreover for $A_i := f_i(B_0, ..., B_i)$ we have $\cap_{i=1}^{\infty} A_i \in S$. We call such a strategy a $(\alpha, \beta; S)$-winning strategy. Thus the n'th move of A corresponds to the function f_{n-1}. (Note that for a given set, there might be several winning strategies and thus several sequences of functions $\{f_i\}_i$.)

Proof. Fix $0 < \beta < 1$. Note that indepedant of A's strategy, in his n'th move B always chooses a ball B_{n-1} of diameter $\beta\rho(A_{n-2})$. Let $S_0, S_1, S_2, ...$ be countably many α-winning sets. Define $S := \cap_{j=1} S_j$. We want to show that S is α-winning.

We first want to make sure that the intersection of sets chosen by A is contained in S_1. For this purpose we assume, that A can only choose her first, third, fifth, ... movement. Assume A made a choice for the set A_0 such that $\rho(A_0) = \alpha\rho(B_0)$. What happens between the first and the third move of A? B chooses a closed ball B_1 of diameter $\beta\rho(A_0)$, then we assume that a closed ball A_1 of diameter $\alpha\rho(B_1)$ is chosen (after some yet unknown rule) and then B chooses again a closed ball B_2 of diameter $\beta\rho(A_1) = \beta\alpha\beta\rho(A_0)$. More generally, if for some odd n A choose a set A_{n-1} then $B_{(n-1)+2}$ has diameter $\beta\alpha\beta\rho(A_n)$. Thus to ensure that $\cap_{n=0}^{\infty} A_n \in S_1$, A should chose a $(\alpha, \beta\alpha\beta; S_1)$-winning strategy.

Now we want to improve the strategy such that $\cap_{n=0}^{\infty} A_n \in S_2$. For the first, third, fifth,... move, A adopts the rule described above and we want to find a rule for the second, sixth, tenth,... move. Using a similar reasoning as above, we find that if A chose a ball A_{n-1} for $n \equiv 2 \quad [4]$, then the diameter of $B_{(n-1)+4}$ is $\beta(\alpha\beta)^3$. Thus A should play according to a $(\alpha, \beta(\alpha\beta)^3; S_2)$-winning strategy.

More generally, if $k \equiv 2^{l-1}[2^l]$ then A plays with a $(\alpha, \beta(\alpha\beta)^{2^l-1}; S_l)$-winning strategy to enforce that $\cap_{n=0}^{\infty} A_n \in S_l$ and thus $\cap_{n=0}^{\infty} A_n \in S$.

Formally we have: For $l = 0, 1, 2, ...$, let $f_0^l, f_1^l, f_2^l ...$ be an $(\alpha, \beta(\alpha\beta)^{2^l-1}; S_2)$-winning strategy.

We want to define a strategy $f_0, f_1, f_2, ...$ for S. Let $k \in \{0, 1, 2, ...\}$. For $k+1 \equiv 2^{l-1}[2^l]$ and $k+1 = 2^{l-1} + (t-1)2^l$ Set

$$A_k := f_k(B_1, ..., B_k) = f_t^l(B_{2^{t-1}}, B_{2^{t-1}+2^l}, ..., B_{2^{t-1}+(t-1)2^l})$$

$\square$

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