Badly approximable numbers

These notes copy Beresnevich "Metric Number Theory" sections 1.5 and 1.6.

In the first lecture, we saw that for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the n-th convergent $\frac{p_n}{q_n}$ satisfies

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}.$$

In particular, we have

Theorem: Dirichlet approximation theorem

For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many integers p and q, $q > 0$, such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

We also saw Hurwitz theorem, which improves the previous result

Theorem: Hurwitz theorem

For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many integers p and q, $q > 0$, such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2}$$

The constant $\frac{1}{\sqrt{5}}$ is the best possible.

This motivates the following definition of badly approximable numbers, which are numbers, whose approximation error by rationals of a given "complexity" q is close to the upper bound in Hurwitz theorem.

Definition:

$\alpha \in \mathbb{R}$ is called badly approximable if there is a constant $c > 0$ such that for all $q, p \in \mathbb{Z}$, $q > 0$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^2}.$$

Recall from last lecture the Lagrange spectrum

$$\mathcal{L} := \{ \nu(\alpha) : \alpha \in \mathbb{R} \}$$

where $\quad \nu(\alpha) = \liminf_{q \to \infty} q \, \|q\alpha\|.$

($\|x\|$ denotes the distance to the nearest integer)

Badly approximable numbers can be characterised in terms of the Lagrange spectrum:

Lemma: $\alpha \in \mathbb{R}$ is badly approximable iff $\nu(\alpha) > 0$.

Proof:

If $\alpha \in \mathbb{R}$ is badly approximable, then for $q \in \mathbb{N}$:

$$q\|q\alpha\| = q|q\alpha - p| \geq c \Rightarrow \nu(\alpha) \geq c > 0.$$

Conversely, $\nu(\alpha) > 0 \Rightarrow \alpha$ irrational and $\exists\, q_0 > 0$ such that for $q > q_0$

$$q\|q\alpha\| > \frac{\nu(\alpha)}{2}.$$

Thus, $\left|\alpha - \frac{p}{q}\right| \geq \frac{\nu(\alpha)}{2q^2}$ for $q > q_0$

and α is badly approximable with

$$c = \min\left(\frac{\nu(\alpha)}{2}, \min_{q \leq q_0} q\|q\alpha\|\right). \quad \square$$

Badly approximable numbers can also be characterised in terms of bounded partial quotients:

<u>Theorem</u>:

A real irrational number $\alpha = [a_0, a_1, \dots]$ is badly approximable iff there is a $C > 0$ such that $a_n \le C$ for all $n \in \mathbb{N}$, i.e. iff the partial quotients are bounded.

Proof: Recall from last lecture that the convergents are best approximations: For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we have

$$\forall q' \in \mathbb{N} \quad q' < q_n \implies \|q'\alpha\| > \|q_n \alpha\|.$$

Thus for any $q \in \mathbb{N}$ with $q_{n-1} < q < q_n$

$$q \|q\alpha\| > q_n \|q_n \alpha\|$$

and $\nu(\alpha) = \liminf\limits_{n \to \infty} q_n \|q_n \alpha\|$.

Furthermore, $\|q_n \alpha\| = |q_n \alpha - p_n|$ for $n > 0$, so that

$$\nu(\alpha) = \liminf_{n \to \infty} q_n \|q_n \alpha\| = \liminf_{n \to \infty} q_n |q_n \alpha - p_n|.$$

From a lemma in the first lecture we have that $q_{n+1} = a_{n+1} q_n + q_{n-1}$ and thus

$$\frac{1}{(a_{n+1}+1)q_n + q_{n-1}} = \frac{1}{q_{n+1}+q_n} < |q_n\alpha - p_n| < \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}}.$$

Furthermore, since $q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1}$,

$$\frac{1}{a_{n+1}+2} < q_n|q_n\alpha - p_n| < \frac{1}{a_{n+1}} \quad \text{so that}$$

$\nu(\alpha) > 0$

$$\Rightarrow \quad 0 < \nu(\alpha) = \liminf_{n\to\infty} q_n|q_n\alpha - p_n|$$

$$< \liminf_{n\to\infty}\left(\frac{1}{a_{n+1}}\right) \quad \Rightarrow \quad a_n \text{ bounded}$$

and conversely,

$$a_n \text{ bounded} \quad \Rightarrow \quad 0 < \liminf_{n\to\infty}\left(\frac{1}{a_{n+1}+2}\right) < \nu(\alpha). \quad \square$$

Corollary: If the continued fraction of a irrational real number α is periodic, α is badly approximable.

As an example of a badly approximable number whose continued fraction expansion is not periodic, consider $\alpha = [1; 1, 2, 1, 1, 2, 1, 1, 1, 2, \dots]$.

Which irrational real numbers have a periodic continued fraction? The Euler - Lagrange theorem answers this.

Definition: $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is called a quadratic irrationality if α is a solution to a quadratic equation

$$ax^2 + bx + c = 0, \quad a, b, c \in \mathbb{Z}, \; a \neq 0$$

Theorem: Euler - Lagrange theorem

Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The continued fraction of α is periodic if and only if α is a quadratic irrationality.

Proof: Suppose $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ with $\alpha = [a_0, \dots, a_{k_0-1}, \overline{a_{k_0}, \dots, a_{k_0+t-1}}]$ and let $\beta = [\overline{a_{k_0}, \dots, a_{k_0+t-1}}]$, so that

$\beta = [a_{k_0}, \dots, a_{k_0+t-1}, \beta]$. From a property of convergents from the first lecture, we have

$$\beta = \frac{p'_{t-1}\beta + p'_{t-2}}{q'_{t-1}\beta + q'_{t-2}}$$ and thus β is a root of a quadratic polynomial with coefficients in $\mathbb{Z}$.

Similarily, since $\alpha = [a_0, \dots, a_{k_0-1}, \beta]$,

$$\alpha = \frac{p_{k_0-1}\beta + p_{k_0-2}}{q_{k_0-1}\beta + q_{k_0-2}}$$

$$\Rightarrow \quad q_{k_0-1}\beta\alpha + q_{k_0-2}\alpha = p_{k_0-1}\beta + p_{k_0-2}$$

$$\Rightarrow \quad \beta(q_{k_0-1}\alpha - p_{k_0-1}) = -q_{k_0-2}\alpha + p_{k_0-2}$$

$$\Rightarrow \quad \beta = \frac{-\alpha q_{k_0-2}\alpha + p_{k_0-2}}{\alpha q_{k_0-1} - p_{k_0-1}}$$, so that

$$f\left(\frac{-x q_{k_0-2} + p_{k_0-2}}{x q_{k_0-1} - p_{k_0-1}}\right)\Bigg|_{x=\alpha} = f(\beta) = 0.$$

Thus $\quad (x q_{n_0-1} - p_{n_0-1})^2 f\left(\dfrac{-x q_{n_0-2} + p_{n_0-2}}{x q_{n_0-1} - p_{n_0-1}}\right)$

is a quadratic polynomial with coefficients in $\mathbb{Z}$,

and α is a quadratic irrationality.

Conversely, let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ be a root of

$f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$, and $\frac{p_n}{q_n}$ the

n-th convergent to α. As recalled at the

beginning, we have $\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$ and thus

by Taylor's formula:

$$\left|f\left(\frac{p_n}{q_n}\right)\right| = \left|0 + f'(\alpha)\left(\frac{p_n}{q_n} - \alpha\right) + \frac{1}{2} f''(\alpha)\left(\frac{p_n}{q_n} - \alpha\right)^2\right|.$$

Since $\left|\frac{p_n}{q_n} - \alpha\right| \xrightarrow{n\to\infty} 0$, we have that

$$\left|f\left(\frac{p_n}{q_n}\right)\right| \leq \left(|f'(\alpha)| + \varepsilon\right)\left|\frac{p_n}{q_n} - \alpha\right| \leq q_n^{-2}\left(|f'(\alpha)| + \varepsilon\right). \qquad (*)$$

Next, we will construct a sequence of quadratic forms $g_n(x,y)$, such that the "tails"

$$\alpha_{n+1} = [a_{n+1}, a_{n+2}, \dots]$$

are roots of $g_n(x,1)$. We will then use the inequality $(*)$ to show that we have only a finite number of distinct quadratic forms g_n.

Thus we have only finitely many tails α_n and there exist $m, n \in \mathbb{N}$ such that $\alpha_{n+m} = \alpha_n$, which implies that the continued fraction of α is periodic.

To construct the sequence consider the quadratic form

$$g(x,y) = ax^2 + bxy + cy^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $g(\alpha, 1) = 0$.

Define a sequence of quadratic forms by

$$g_n(x,y) = a_n x^2 + b_n xy + c_n y^2$$

$$= \begin{pmatrix} x & y \end{pmatrix} T_n^t \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} T_n \begin{pmatrix} x \\ y \end{pmatrix}$$

where $T_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$, and let

$\alpha_{n+1} := [a_{n+1}, a_{n+2}, \ldots]$, so that

$$\alpha = [a_1, \ldots, a_n, \alpha_{n+1}] \Rightarrow \alpha = \frac{p_n \alpha_{n+1} + p_{n-1}}{q_n \alpha_{n+1} + q_{n-1}}.$$

Since $g_n(x,y) = g(p_n x + p_{n-1} y,\ q_n x + q_{n-1} y)$

the above implies

$$g_n(\alpha_{n+1}, 1) = g(p_n \alpha_{n+1} + p_{n-1},\ q_n \alpha_{n+1} + q_{n-1})$$

$$= g(\alpha(q_n \alpha_{n+1} + q_{n-1}),\ q_n \alpha_{n+1} + q_{n-1})$$

$$= g(\alpha, 1) = f(\alpha) = 0.$$

Furthermore, $a_n = g_n(1,0) = g(p_n, q_n)$

$\Rightarrow$ $c_n = g_n(0,1) = g(p_{n-1}, q_{n-1}) = a_{n-1}$

Since $g(p_n, q_n) = a p_n^2 + b p_n q_n + c q_n^2$

$$= q_n^2 \left(a \frac{p_n^2}{q_n^2} + b \frac{p_n}{q_n} + c \right)$$

$= q_n^2 f\left(\frac{p_n}{q_n}\right)$, the inequality

$(*)$ $\Rightarrow$ $|a_n| \leq |f'(\alpha)| + \varepsilon$

$\Rightarrow$ finitely many a_n and $c_n = a_{n-1}$.

Finally, we have $\det T_n = p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$ from an earlier lecture and thus

$$a_n c_n - \frac{b_n^2}{4} = \det T_n^t \begin{pmatrix} a & \frac{1}{2} b \\ \frac{1}{2} b & c \end{pmatrix} T_n = ac - \frac{b^2}{4}.$$

Thus there are only finitely many b_n.
This concludes the proof. $\square$

As a corollary we get

<u>Theorem</u>: Any quadratic irrationality is badly approximable.

It is not known whether real algebraic numbers of degree ≥ 3 are badly approximable.

References

1. V. Beresnevich, Metric Number Theory, Lecture notes, University of York, 2014.

2. V. Beresnevich, Number Theory, Lecture notes, University of York, 2013.