Continued fractions

Note: These notes are based on the lecture notes of Victor Beresnevich "Number Theory", chapters 3.4, 3.5 (aviable on the homepage of the seminary)
You will find there the complete proofs.

Def: A finite continued fraction is an expression of the following form
$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}}$$

denoted by $[a_0; a_1, \dots, a_n]$ where $a_i \in \mathbb{R}$ $\forall i \geq 0$

Example: $[1;2,3,4] = 1 + \cfrac{1}{2+\cfrac{1}{3+\frac{1}{4}}} = 1 + \cfrac{1}{2+\frac{4}{13}} = 1 + \frac{13}{30} = \frac{43}{30}$

Def: An infinite continued fraction is an expression of the form $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}$

that is understood as an infinite sequence of finite continued fractions

Question: We know how to calculate a number starting from a continued fraction.
But how can we find the continued fraction of some real number?

Example: $\frac{13}{5} \underset{\substack{\uparrow \\ \text{Take the integer part}}}{=} 2 + \frac{3}{5} \underset{\substack{\uparrow \\ \text{Invert the fraction}}}{=} 2 + \cfrac{1}{\frac{5}{3}} \underset{\substack{\uparrow \\ \text{repeat}}}{=} 2 + \cfrac{1}{1+\frac{2}{3}} = \dots$

In general, there exists an algorithm

The continued fraction algorithm

Let $x \in \mathbb{R}$

1) Let $x_0 = x$ and $a_0 = [x_0]$ ← integer part

2) For $k = 0, 1, 2, \ldots$ if $x_k \neq a_k$ define $x_{k+1} = \dfrac{1}{x_k - a_k}$

and $a_{k+1} = [x_{k+1}]$

If $x_k = a_k$, the algorithm terminate

Note: If the algorithm terminates, we obtain a finite contined fraction, otherwise is infinite

Why does it work?

$$x_{k+1} = \frac{1}{x_k - a_k} \Rightarrow x_k = a_k + \frac{1}{x_{k+1}}$$

$$\text{So } x = x_0 = a_0 + \frac{1}{x_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}} = \ldots$$

Example of before: $\frac{13}{5} = x_0 \qquad a_0 = \left[\frac{13}{5}\right] = 2$

$$x_1 = \frac{1}{\frac{13}{5} - 2} = \frac{1}{\frac{3}{5}} = \frac{5}{3} \qquad a_1 = \left[\frac{5}{3}\right] = 1$$

$$x_2 = \frac{1}{\frac{5}{3} - 1} = \frac{1}{\frac{2}{3}} = \frac{3}{2} \qquad a_2 = \left[\frac{3}{2}\right] = 1$$

$$x_3 = \frac{1}{\frac{3}{2} - 1} = \frac{1}{\frac{1}{2}} = 2 \qquad a_3 = 2$$

$$\Rightarrow \frac{13}{5} = [2; 1, 1, 2]$$

Properties of the algorithm:

1) $x_k - a_k = x_k - [x_k] = \{x_k\} < 1$ and $0 \leq \{x_k\}$

↖ fractional part

$\Rightarrow x_{k+1} = \dfrac{1}{x_k - a_k} > 1$ (if $x_k \neq a_k$)

$\Rightarrow a_0 \in \mathbb{Z}$ and $a_1, a_2, \ldots \in \mathbb{Z}_{>0}$

2) As said before, the recursive formula gives

$$x = a_0 + \frac{1}{x_1} = a_0 + \frac{1}{a_1 + \frac{1}{x_2}} = \dots$$

If we stop the algorithm arbitrary after n steps we obtain the finite continued fraction

$$x = [\underbrace{a_0, a_1, a_2, \dots, a_{n-1}}_{\in \mathbb{Z}}, \underbrace{x_n}_{\in \mathbb{R}}]$$

(This will be very helpful in some proofs)

3) $x \in \mathbb{Q} \Leftrightarrow$ The continued fraction of x (given by the algorithm) is finite

Proof: "$\Leftarrow$" Clear: we have a fraction of integers

"$\Rightarrow$": Let $\frac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{Z}$ coprimes

Using the algorithm we obtain

$$x_0 = \frac{p}{q} \qquad a_0 = \left[\frac{p}{q}\right]$$

$$x_0 - a_0 = \frac{p'}{q} \quad \text{wher } p' < q$$

$$x_1 = \frac{q}{p'} \qquad a_1 = [x_1]$$

$$x_1 - a_1 = \frac{q'}{p'} \quad \text{where } q' < p'$$

So at every step p or q become smaller

So the algorithm must terminates in less than $\max\{p, q\}$ steps

$\Rightarrow$ finite continued fraction $\blacksquare$

Example: Continued fraction of $\sqrt{2}$

$$x_0 = \sqrt{2} \qquad a_0 = [\sqrt{2}] = 1$$

$$x_1 = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{1} \qquad a_1 = [\sqrt{2}+1] = 2$$

$x_2 = \frac{1}{\sqrt{2}+1-2} = \frac{1}{\sqrt{2}-1} = x_1$, we are in a loop $\Rightarrow \sqrt{2} = [1; 2, 2, 2, 2, \dots]$

• Gold ratio: $\frac{\sqrt{5}+1}{2} = [1;1,1,1,\dots]$

Def: Given a continued fraction $[a_0; a_1, \dots]$ (finite or infinite) the finite subfraction $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$ is called its nth convergent and the numbers a_n are called partial quotients.

You can calculate p_n, q_n directly or using:

Lemma: Let $n \geq 0$ and

$$p_0 = a_0 \qquad q_0 = 1$$
$$p_1 = a_1 a_0 + 1 \qquad q_1 = a_1$$
$$p_k = a_k p_{k-1} + p_{k-2} \qquad q_k = a_k q_{k-1} + q_{k-2} \qquad \text{for } 2 \leq k \leq n$$

Then $\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$

Proof: We use induction on n.
The cases $n = 0, 1, 2$ are just computation ✓
Induction step: Assuming the formula works for $a_i > 0$ and $m \leq n$

Consider $[a_0; a_1, \dots, a_{n+1}] \underset{\uparrow}{=} [a_0; a_1, \dots, a_{n-1}, a_n + \frac{1}{a_{n+1}}]$
By definition

Using induction we get $[a_0; a_1, \dots, a_n, a_{n+1}] = \frac{p_{n+1}}{q_{n+1}}$

Example: Let the continued fraction $[3; 7, 15, 1, \dots]$

$p_0 = 3$	$q_0 = 1$	$\frac{p_0}{q_0} = \frac{3}{1}$
$p_1 = 22$	$q_1 = 7$	$\frac{p_1}{q_1} = \frac{22}{7} \cong 3{,}14285$
$p_2 = 15 \cdot 22 + 3 = 333$	$q_2 = 15 \cdot 7 + 1 = 106$	$\frac{p_2}{q_2} = \frac{333}{106} \cong 3{,}141509$
$p_3 = 1 \cdot 333 + 22 = 355$	$q_3 = 1 \cdot 106 + 7 = 113$	$\frac{p_3}{q_3} = \frac{355}{113} \cong 3{,}1415929$

As you can imagine, this is the continued fraction of π.

Note: With this example we see that with the increasing of n, the nth convergent become a better approximation of the number that is representated by the continued fraction.
In the next chapter, we will see these convergence properties.

Convergence properties of continued fractions.

Lemma: Let p_n, q_n given by the previous formula, then for $n \geq 1$

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}$$

Proof: Use induction; very straightforward ■

Corollary: Let $a_0, \dots, a_n \in \mathbb{Z}$ with $a_1, \dots, a_n$ positive, then $\gcd(p_n, q_n) = 1$

Lemma: Using x_i, a_i as in the algorithm of continued fractions we have $\forall x \in \mathbb{R}$ and $\forall n \geq 2$

$$x = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}}$$

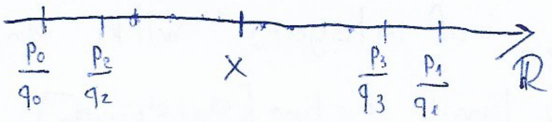
Proof: $x = [a_0, \dots, a_{n-1}, x_n] = \frac{p_n}{q_n} = \frac{x_n p_{n-1} + p_{n-2}}{x_n q_{n-1} + q_{n-2}}$ ■

Lemma: Let $x \in \mathbb{R}$ and $[a_0, a_1, \dots]$ the continued fraction obtained by the algorithm, then

$$x \leq \frac{p_n}{q_n} \quad \text{if } n \text{ is odd}$$

$$x \geq \frac{p_n}{q_n} \quad \text{if } n \text{ is even}$$

Furthermore, if $n \geq 2$ $\frac{p_n}{q_n}$ is between $\frac{p_{n-1}}{q_{n-1}}$ and $\frac{p_{n-2}}{q_{n-2}}$



Proof: Induction

For $n=0$ and $n=1$ use that $a_i = [x_i] \leq x_i$ ✓

For induction step consider the function $f(t) = \frac{t p_{n-1} + p_{n-2}}{t q_{n-1} + q_{n-2}}$

We know $f(0) = \frac{p_{n-2}}{q_{n-2}}$, $\underset{\substack{\uparrow \\ \text{formula}}}{f(a_n) = \frac{p_n}{q_n}}$, $\underset{\substack{\uparrow \\ \text{previous lemma}}}{f(x_n) = x}$, $f(\infty) = \frac{p_{n-1}}{q_{n-1}}$

Calculating the derivative: $f'(t) = \frac{p_{n-1} q_{n-2} - q_{n-1} p_{n-2}}{(t q_{n-1} + q_{n-2})^2} = \frac{(-1)^n}{(t q_{n-1} + q_{n-2})^2}$

This means that f is increasing for n even and decresing for n odd

Using $0 < a_n \leq x_n < \infty$ we get the result ■

Lemma: Let $x \in \mathbb{R}$ and $[a_0; a_1, \dots]$ the continued fraction obtained from the algorithm, then for any $n \geq 0$

$$\frac{1}{q_n(q_{n+1}+q_n)} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

In particular $\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$ (because $q_{n+1} \geq q_n$)

Proof: Using previous lemma we have

That means

$$\underbrace{\left| \frac{p_n}{q_n} - \frac{p_{n+2}}{q_{n+2}} \right|}_{\geq \frac{a_{n+2}}{q_n q_{n+2}} \geq \frac{a_{n+2}}{q_n(a_{n+2}q_{n+1}+q_n)} > \frac{1}{q_n(q_{n+1}+q_n)}} \leq \left| x - \frac{p_n}{q_n} \right| < \underbrace{\left| \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} \right|}_{= \frac{1}{q_n q_{n+1}}}$$

$\blacksquare$

Note: Comparing the bounds we get that $\frac{p_{n+1}}{q_{n+1}}$ will be closer to x than $\frac{p_n}{q_n}$

Corollary: If $x \notin \mathbb{Q}$, then $x = \lim_{n\to\infty} \frac{p_n}{q_n} = \lim_{n\to\infty} [a_0; a_1, \dots, a_n]$

(In fact q_n increases of at least 1 in every step, so $\frac{1}{q_n^2} \to 0$ as $n \to \infty$).

Lemma: For any sequence $(a_n)_{n\geq 0}$ of integers with $a_n \geq 1$ for $n \geq 1$, there exists the limit $x = \lim_{n\to\infty} [a_0; a_1, \dots, a_n]$

Furthermore, the infinite continued fraction of x, obtained by the algorithm, will be exactly $[a_0; a_1, \dots]$

Proof: • Convergence:

$$q_n = a_n q_{n-1} + q_{n-2} \underset{\substack{\uparrow \\ a_n \geq 1}}{\geq} 2q_{n-2} \overset{q_0=1}{\geq} 2^{(n-1)/2}$$

Let $m \leq n$: $\left|\frac{p_m}{q_m} - \frac{p_{n+1}}{q_{n+1}}\right| = \left|\sum_{k=m}^{n}\left(\frac{p_k}{q_k} - \frac{p_{k+1}}{q_{k+1}}\right)\right| = \left|\sum_{k=m}^{n} \frac{(-1)^{k+1}}{q_k q_{k+1}}\right| \leq$

$$\leq \sum_{k=m}^{\infty} \frac{1}{q_k^2} < \sum_{k=m}^{\infty} \frac{1}{2^{k-1}} = \frac{2^{-m+1}}{1-\frac{1}{2}} = 2^{-m+2} \xrightarrow[m\to\infty]{} 0$$

$\Rightarrow \frac{p_n}{q_n}$ is a Cauchy sequence and so converges

• Uniqueness:

Suppose x has two different sequences $x = [a_0, a_1, \dots] = [a_0', a_1', \dots]$

Let k minimal st $a_k \neq a_k'$. Suppose $a_k' \geq a_k + 1$

Let $x_k = [a_k, a_{k+1}, \dots]$, $x_k' = [a_k', a_{k+1}', \dots]$

$\Rightarrow x_k = x_k'$ ("finite fractions")

$$x_k' = a_k' + \frac{1}{[a_{k+1}', \dots]} > a_k' \geq a_k + 1 \geq a_k + \frac{1}{[a_{k+1}, \dots]} = x_k \quad \lightning \quad \blacksquare$$

This lemma says that the algorithm of continued fractions is a bijection between irrational numbers and infinite sequences $(a_n)_{n\geq 0}$ of integers with $a_1, a_2, \dots > 0$