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Markov's Problem
from a Hyperbolic Geometry point of view

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Quadratic forms, Markov numbers and Diophantine approximation

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Introduction

This paper covers most of the material of sections 10–13 of [Spr17]. We start by assigning a geodesic to each indefinite binary quadratic form and establishing a result about the signed distance of a horocycle and a geodesic. Afterwards, we turn towards hyperbolic triangles whose vertices are ideal points and introduce the modular torus. After having established a connection between ideal triangulations of the modular torus and Markov triples, we present the solution and an application of a geometric optimization problem.

We refer to [Yan20] for the definitions of objects like isometries of the hyperbolic plane, hyperbolic distance or horocycles. A detailed discussion of the isometries of the hyperbolic plane and the hyperbolic distance can be found in [And05].

The majority of the figures of this paper have been created using GeoGebra (see[Arn]).

I would like to thank Dr. Paloma Bengoechea for supervising this paper and especially for her helpful explanations concerning the proof of Proposition 12.1 of [Spr17].

1 Geodesics and indefinite binary quadratic forms

Definition 1.1
We assign to every indefinite binary quadratic form $f(x,y) = ax^2 + bxy + cy^2$ with a, b and $c \in \mathbb{R}$ the geodesic connecting the zeros of the polynomial $f(x,1) = ax^2 + bx + c$ and call this geodesic $g(f)$.

Remark
- *If $a \neq 0$, then $f(x,1)$ has two distinct real roots $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ because $b^2 - 4ac > 0$ and $g(f)$ is a euclidean half circle with origin on the real axis.*
- *If $a = 0$, then we consider $-\frac{c}{b}$ and ∞ as two roots of $f(x,1)$ and $g(f)$ is a vertical euclidean line.*

Example
Let $f(x,y) = x^2 + 4xy + 3y^2$. f is an indefinite quadratic form because $disc(f) = 16 - 12 = 4 > 0$. The zeros of the polynomial $f(x,1) = x^2 + 4x + 3$ are -3 and -1. Therefore, $g(f)$ is the euclidean half circle with origin -2 connecting -3 and -1.

Lemma 1.1
The map g that assigns to each indefinite binary quadratic form f the geodesic $g(f)$ is surjective and many-to-one. It holds:

$$g(f_1) = g(f_2) \iff f_1 = \alpha f_2 \text{ for some } \alpha \in \mathbb{R} \setminus \{0\}.$$

Lemma 1.2
If $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{R})$ and $f(x,y) = ax^2 + bxy + cy^2$ is an indefinite binary quadratic form, then

$$g(f \circ A^{-1}) = M_A(g(f)),$$

i.e. the geodesic assigned to the binary quadratic form

$$(f \circ A^{-1})(x,y) = f\left(\frac{\delta x - \beta y}{\alpha\delta - \beta\gamma}, \frac{-\gamma x + \alpha y}{\alpha\delta - \beta\gamma}\right)$$

is the image of $g(f)$ under the isometry

$$M_A(z) = \frac{\alpha z + \beta}{\gamma z + \delta}.$$

Proof. First, we determine the zeros of the polynomial $(f \circ A^{-1})(x,1)$.

$$\begin{aligned}
(f \circ A^{-1})(x,y) &= f\left(\frac{\delta x - \beta y}{\alpha\delta - \beta\gamma}, \frac{-\gamma x + \alpha y}{\alpha\delta - \beta\gamma}\right) \\
&= a\left(\frac{\delta x - \beta y}{\alpha\delta - \beta\gamma}\right)^2 + b\frac{\delta x - \beta y}{\alpha\delta - \beta\gamma}\frac{-\gamma x + \alpha y}{\alpha\delta - \beta\gamma} + c\left(\frac{-\gamma x + \alpha y}{\alpha\delta - \beta\gamma}\right)^2 \\
&= \frac{(a\delta^2 - b\gamma\delta + c\gamma^2)x^2 + (b\alpha\delta + b\beta\gamma - 2a\beta\delta - 2c\alpha\gamma)xy + (a\beta^2 - b\alpha\beta + c\alpha^2)y^2}{(\alpha\delta - \beta\gamma)^2}
\end{aligned}$$

This implies that the zeros of $(f \circ A^{-1})(x,1)$ are

$$y_{1,2} = \frac{2a\beta\delta + 2c\alpha\gamma - b\alpha\delta - b\beta\gamma \pm (\alpha\delta - \beta\gamma)\sqrt{b^2 - 4ac}}{2(a\delta^2 - b\gamma\delta + c\gamma^2)}$$

Now, we calculate the image of the zeros

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

of $f(x,1)$ under M_A.

$$\begin{aligned} M_A(x_i) &= \frac{\alpha x_i + \beta}{\gamma x_i + \delta} \\ &= \frac{(-1)^{i+1}\alpha\sqrt{b^2-4ac} - b\alpha + 2a\beta}{(-1)^{i+1}\gamma\sqrt{b^2-4ac} - b\gamma + 2a\delta} \\ &= \frac{2a\beta\delta + 2c\alpha\gamma - b\alpha\delta - b\beta\gamma + (-1)^{i+1}(\alpha\delta - \beta\gamma)\sqrt{b^2-4ac}}{2(a\delta^2 - b\gamma\delta + c\gamma^2)} \\ &= y_i \quad \forall\, i \in \{1,2\}. \end{aligned}$$

Since the image of a geodesic under M_A is a geodesic, the result follows.

□

We refer to Definition 1.9 of [Yan20] for the definition of the signed distance between a geodesic and a horocycle.

Proposition 1.1
Let $f = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form and let $(p,q) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
The signed distance of the horocycle $h(p,q)$ and the geodesic $g(f)$ is

$$d(h(p,q), g(f)) = \log \frac{2|f(p,q)|}{\sqrt{disc(f)}}. \tag{1}$$

Proof. First, we prove the result for some special cases. Afterwards we reduce the general case to these special cases. We will use this strategy to prove several other results.

1. If $q = 0$ and $a = 0$, i.e. $h(p,q)$ is the horocycle at ∞ at height p^2 and $g(f)$ is a euclidean vertical line: Since $g(f)$ ends in the center of $h(p,q)$, we have:

$$\log \frac{2|f(p,q)|}{\sqrt{\mathrm{disc}(f)}} = \log(0) = d(h(p,q), g(f)) = -\infty$$

   because $f(p,q) = q(bp + cq) = 0$.

2. If $q = 0$ and $a \neq 0$, i.e. $h(p,q)$ is the horocycle at ∞ at height p^2 and $g(f)$ is a euclidean half circle with origin on the real axis:
   We have:

$$\delta := \frac{1}{2}\left(\frac{-b + \sqrt{b^2-4ac}}{2a} - \frac{-b - \sqrt{b^2-4ac}}{2a}\right) = \frac{\sqrt{b^2-4ac}}{2a} = \frac{p^2\sqrt{\mathrm{disc}(f)}}{2|f(p,q)|}$$

   is half of the euclidean distance between the two zeros $x_{1,2}$ of $f(x,1)$.
   Since $d(h(p,q), g(f))$ is the distance of the point $\frac{x_1+x_2}{2} + \delta i$ to $h(p,q)$ taken positive if $p^2 \geq \delta$ and taken negative if $p^2 < \delta$ we get:

$$d(h(p,q), g(f)) = \log \frac{p^2}{\delta} = \log \frac{2|f(p,q)|}{\sqrt{\mathrm{disc}(f)}}$$

3. If $q \neq 0$, i.e. $h(p,q)$ is the horocycle at $\frac{p}{q}$ with euclidean diameter $\frac{1}{q^2}$:
   Define $A = \begin{pmatrix} p & \frac{1-p^2}{q} \\ -q & p \end{pmatrix}$.
   It holds:
   (a) $\det(A) = 1$
   (b) $A\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

By the second special case, we get:

$$\begin{aligned}d(h(p,q),g(f)) = d(M_A(h(p,q)),M_A(g(f)) &= d(h(1,0),g(f\circ A^{-1})\\ &= \log\frac{2|(f\circ A^{-1})(1,0)|}{\sqrt{\operatorname{disc}(f\circ A^{-1})}} = \log\frac{2|ap^2+bpq+cq^2|}{\operatorname{disc}(f)}\\ &= \log\frac{2|f(p,q)|}{\sqrt{\operatorname{disc}(f)}}\end{aligned}$$

□

Lemma 1.3
If $f(x,y)=x^2-xy-y^2$ and $(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}$ is such that $h(p,q)$ and $g(f)$ intersect, then the signed distance of $h(p,q)$ and $g(f)$ satisfies

$$d(h(p,q),g(f)) = \log(\frac{2}{\sqrt{5}}).$$

Proof. Define $A=\begin{pmatrix} p & \frac{1-p^2}{q}\\ -q & p\end{pmatrix}$.

We have:

$$z_{1,2} := M_A(\frac{1\pm\sqrt{5}}{2}) = \text{zeros of } (f\circ A^{-1})(x,1) = \frac{2(1-p^2)\frac{p}{q}+2pq+2p^2-1\pm\sqrt{5}}{2(p^2-pq+q^2)}$$

Furthermore:

- The image of $g(f)$ under M_A is the geodesic connecting z_1 and z_2, i.e. the euclidean half circle with origin $m := \frac{z_1+z_2}{2}$ and radius $r := \frac{|z_1-z_2|}{2}$.
- The image of the horocycle $h(p,q)$ under M_A is the horocycle at ∞ at height 1.

Since $g(f)$ and $h(p,q)$ intersect, $M_A(g(f))$ and $h(1,0)$ intersect as well.

Therefore:

$$r = \frac{\sqrt{5}}{2|p^2-pq+q^2|}\geq 1$$

which implies that

$$|p^2-pq+q^2| = 1.$$

We conclude:

$$d(h(p,q),g(f)) = d(h(1,0),M_A(g(f))) = -d(m+i,m+ri) = -\log(r) = \log(\frac{2}{\sqrt{5}})$$

□

Corollary 1.1
If $\tilde{f}$ is an indefinite binary quadratic form with real coefficients that is equivalent to $f(x,y)=x^2-xy-y^2$ and $(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}$ is such that $h(p,q)$ and $g(\tilde{f})$ intersect, then the signed distance of $h(p,q)$ and $g(\tilde{f})$ satisfies

$$d(h(p,q),g(\tilde{f})) = \log(\frac{2}{\sqrt{5}}).$$

Proof. Let $a,b,c,d\in\mathbb{Z}$ be such that $\tilde{f}(ax+by,cx+dy)=f(x,y)$ and $|ad-bc|=1$.

We have:

$$f(x,y) = (\tilde{f}\circ A^{-1})(x,y) \text{ where } A=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}.$$

Since $h(p,q)$ and $g(\tilde{f})$ intersect, $M_A(h(p,q)$ and $M_A(g(\tilde{f}))$ intersect as well.

By Lemma 1.2 and Lemma 1.3 it follows:

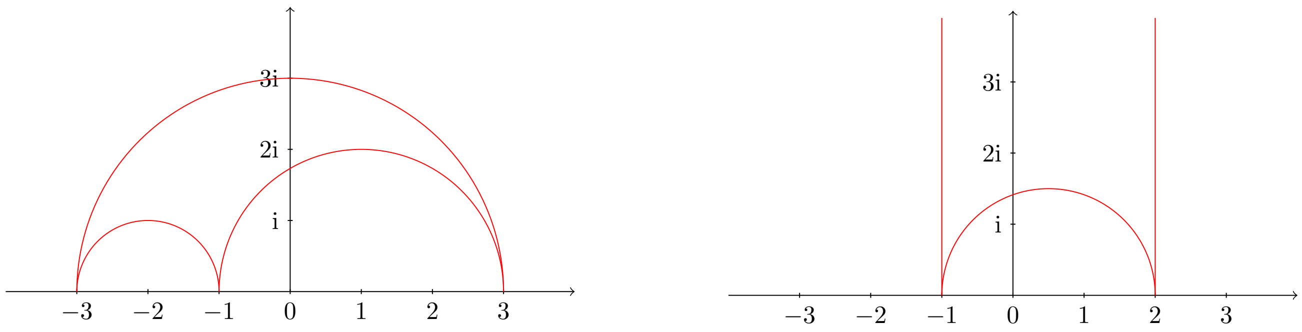
$$d(h(p,q),g(\tilde{f})) = d(M_A(h(p,q),M_A(g(\tilde{f}))) = d(M_A(h(p,q),g(f)) = \log(\frac{2}{\sqrt{5}})$$

□

2 Decorated Ideal Triangles

Definition 2.1
An ideal triangle is a closed region in the hyperbolic plane that is bounded by three geodesics (called the sides) connecting three ideal points v_1, v_2 and v_3 (called the vertices). We denote it by $\mathcal{T}(v_1, v_2, v_3)$.

Example (Ideal triangles)



Lemma 2.1
For each ideal triangle $T = \mathcal{T}(v_1, v_2, v_3)$ there is a hyperbolic isometry M_T that maps T to $\mathcal{T}(0, 1, \infty)$.

Proof. 1. If $v_1 = \infty$: Define

$$M_T(z) = \frac{v_2 - v_3}{z - v_3}$$

2. If $v_2 = \infty$: Define

$$M_T(z) = \frac{z - v_1}{z - v_3}$$

3. If $v_3 = \infty$: Define

$$M_T(z) = \frac{z - v_1}{v_2 - v_1}$$

4. Otherwise: Define

$$M_T(z) = \frac{(z - v_1)(v_2 - v_3)}{(z - v_3)(v_2 - v_1)}$$

□

Corollary 2.1
For any two ideal triangles $T_1 = \mathcal{T}(v_1, v_2, v_3)$ and $T_2 = \mathcal{T}(w_1, w_2, w_3)$ there exists a hyperbolic isometry M_{T_1,T_2} such that M_{T_1,T_2} maps T_1 to T_2.

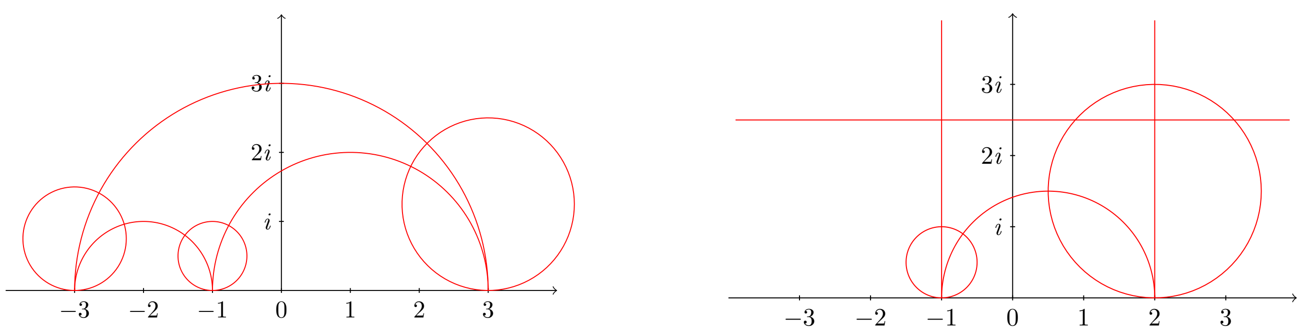
Proof. Define

$$M_{T_1,T_2} = (M_{T_2})^{-1} \circ M_{T_1}$$

□

Definition 2.2
A decorated ideal triangle is an ideal triangle together with a horocycle at each vertex.

Example (Decorated ideal triangles)



Definition 2.3 *1. A decorated geodesic is a geodesic together with a horocycle at each end.*

2. *The truncated length α of a decorated geodesic is the signed distance of its two horocycles $h_1 = h(p_1, q_1)$ and $h_2 = h(p_2, q_2)$, i.e. $\alpha = 2\log|p_1q_2 - p_2q_1|$ (see Definition 1.7 of [Yan20]).*

3. *The weight of a decorated geodesic is defined as $a = e^{\alpha/2}$, where α is the truncated length of the decorated geodesic.*

Remark (Notation)
We write the truncated lengths of the sides of a decorated triangle as a triple $(\alpha_1, \alpha_2, \alpha_3)$ where α_i is the truncated length of the decorated geodesic consisting of the side of the decorated triangle connecting the vertices v_j and v_k together with the horocycles at the vertices v_j and v_k, $\{j,k\} = \{1,2,3\} \setminus \{i\}$.
Similarly, we write (a_1, a_2, a_3) for the weights of a decorated ideal triangle and use the same letters for describing the sides of a triangle and their weights.

Lemma 2.2
Any triple $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ determines a decorated triangle with truncated lengths $(\alpha_1, \alpha_2, \alpha_3)$ that is unique up to isometry.

Proof. First, we consider the decorated triangle with vertices 0, 1, and ∞ and horocycles $h_1 = h(0, A)$, $h_2 = h(B, B)$ and $h_3 = h(C, 0)$, where

$$A = \exp(\frac{\alpha_2 + \alpha_3 - \alpha_1}{4}), B = \exp(\frac{\alpha_1 + \alpha_3 - \alpha_2}{4}) \text{ and } C = \exp(\frac{\alpha_1 + \alpha_2 - \alpha_3}{4}).$$

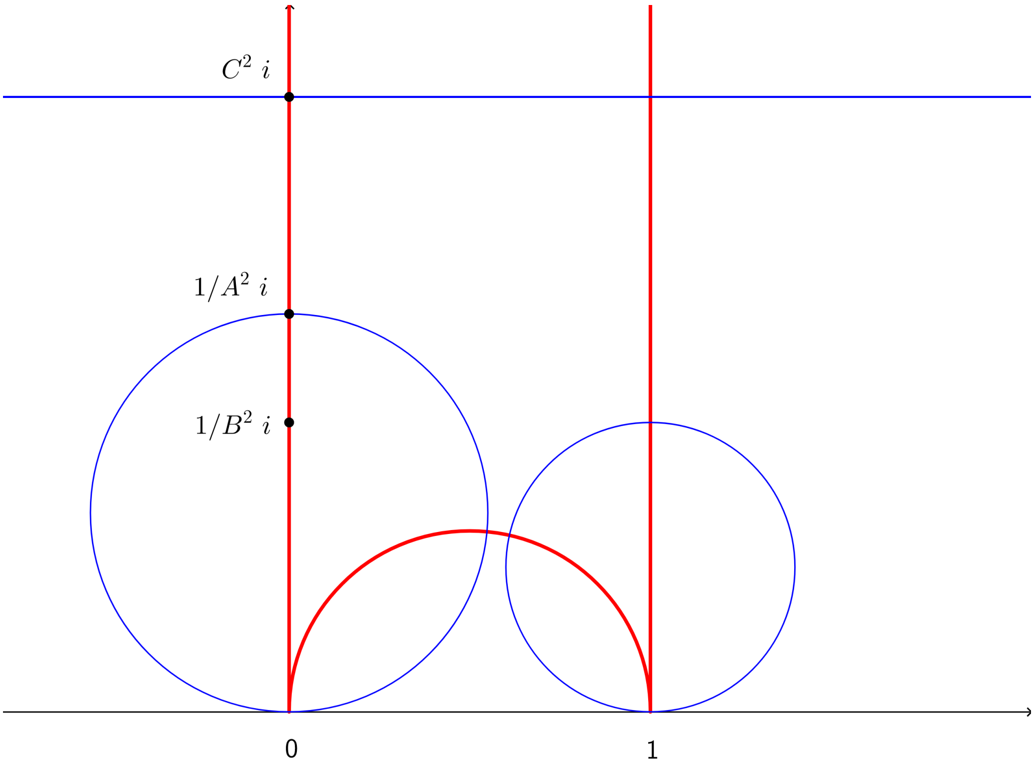


Figure 1: Decorated ideal triangle

We have:

$$d(h_2, h_3) = 2\log(BC) = 2\log(\exp(\frac{\alpha_1}{2})) = \alpha_1$$
$$d(h_1, h_3) = 2\log(AC) = 2\log(\exp(\frac{\alpha_2}{2})) = \alpha_2$$
$$d(h_1, h_2) = 2\log(AB) = 2\log(\exp(\frac{\alpha_3}{2})) = \alpha_3$$

Now, let us consider an arbitrary decorated ideal triangle with vertices v_1, v_2 and v_3 and horocycles $h_1 = h(p_1, q_1)$, $h_2 = h(p_2, q_2)$ and $h_3 = h(p_3, q_3)$ whose truncated lengths are $(\alpha_1, \alpha_2, \alpha_3)$.

Let M be the isometry we defined in Lemma 2.1 that maps the triangle $\mathcal{T}(v_1, v_2, v_2)$ to the triangle $\mathcal{T}(0, 1, \infty)$.

If $v_1 = \infty$, then $q_1 = 0, v_2 = \frac{p_2}{q_2}$, $v_3 = \frac{p_3}{q_3}$ and M is induced by the matrix

$$A = \begin{pmatrix} 0 & \frac{v_2 - v_3}{\delta} \\ \frac{1}{\delta} & -\frac{v_3}{\delta} \end{pmatrix}, \text{ where } \delta = \sqrt{|v_2 - v_3|}.$$

Hence, by Lemma 1.6 of [Yan20], M maps

- h_1 to $h(0, \frac{p_1}{\delta})$,
- h_2 to $h(\frac{q_2(v_2-v_3)}{\delta}), \frac{q_2(v_2-v_3)}{\delta})$ and
- h_3 to $h(\frac{p_3(v_2-v_3)}{\delta}, 0)$.

Since the truncated length of a decorated geodesic is invariant under hyperbolic isometries:

$$\alpha_1 = d(M(h_2), M(h_3)) = 2\log \frac{|p_3 q_2|(v_2 - v_3)^2}{\delta^2} = 2\log(BC)$$
$$\alpha_2 = d(M(h_1), M(h_3)) = 2\log \frac{|p_1 p_3(v_2 - v_3)|}{\delta^2} = 2\log(AC)$$
$$\alpha_3 = d(M(h_1), M(h_2)) = 2\log \frac{|p_1 q_2(v_2 - v_3)|}{\delta^2} = 2\log(AB)$$

Therefore,

$$\frac{p_1}{\delta} \in \{\pm A\},\ \frac{q_2(v_2 - v_3)}{\delta} \in \{\pm B\} \text{ and } \frac{p_3(v_2 - v_3)}{\delta} \in \{\pm C\}$$

which implies that

$$M(h_1) = h(A, 0),\ M(h_2) = h(B, B) \text{ and } M(h_3) = h(0, C).$$

The cases $v_2 = \infty$, $v_3 = \infty$ and $v_1, v_2, v_3 \in \mathbb{R}$ follow analogously. □

Corollary 2.2
Any triple $(a_1, a_2, a_3) \in \mathbb{R}^3_{>0}$ determines a decorated triangle with weights (a_1, a_2, a_3) that is unique up to isometry.

Corollary 2.3
Each decorated ideal triangle that is part of the Farey tessellation together with the Ford circles is isometric to the decorated ideal triangle with vertices 0, 1 and ∞ and horocycles $h_1 = h(0, 1)$, $h_2 = h(1, 1)$ and $h_3 = h(1, 0)$.

Proof. The statement follows from the proof of Lemma 2.2 and the fact that each such decorated ideal triangle has truncated lengths $(0, 0, 0)$. □

Definition 2.4
We consider a decorated ideal triangle. Its three horocycles intersect the triangle in three arcs. We denote the hyperbolic length of the intersection of the horocycle at vertex i with the triangle by c_i and refer to these lengths as horocyclic arc lengths.

Lemma 2.3
The truncated side lengths $(\alpha_1, \alpha_2, \alpha_3)$ of a decorated ideal triangle determine the horocyclic arc lengths (c_1, c_2, c_3), and vice versa, via the relation

$$c_i = \frac{a_i}{a_j a_k} = e^{\frac{1}{2}(-\alpha_i - \alpha_j + \alpha_k)}$$

where (i, j, k) is a permutation of $(1, 2, 3)$.

Proof. We consider the decorated triangle with vertices 0, 1, and ∞ and horocycles $h_1 = h(0, A)$, $h_2 = h(B, B)$ and $h_3 = h(C, 0)$, where

$$A = \exp(\frac{\alpha_2 + \alpha_3 - \alpha_1}{4}), B = \exp(\frac{\alpha_1 + \alpha_3 - \alpha_2}{4}) \text{ and } C = \exp(\frac{\alpha_1 + \alpha_2 - \alpha_3}{4}).$$

Recall that this decorated ideal triangle has truncated side lengths $(\alpha_1, \alpha_2, \alpha_3)$.

As all decorated ideal triangles with truncated lengths $(\alpha_1, \alpha_2, \alpha_3)$ are isometric to the decorated ideal triangle we described above (see Lemma 2.2) and the length of a curve is invariant under isometries, it suffices to prove the result for this special case.

It holds:

$$c_3 = \int_0^1 \frac{1}{C^2} dx = \frac{1}{C^2} = e^{\frac{1}{2}(-\alpha_1 - \alpha_2 + \alpha_3)} = \frac{a_3}{a_1 a_2}$$

By Lemma 1.6 of [Yan20], the isometry $M_{\mathcal{A}}$ induced by the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

maps

- 0 to 1
- 1 to ∞
- ∞ to 0

- $h(0, A)$ to $h(A, A)$
- $h(B, B)$ to $h(B, 0)$
- $h(C, 0)$ to $h(0, -C) = h(0, C)$

Since isometries of the hyperbolic plane preserve the length of a curve, we get

$$c_2 = \int_0^1 \frac{1}{B^2} dx = \frac{1}{B^2} = e^{\frac{1}{2}(-\alpha_1 + \alpha_2 - \alpha_3)} = \frac{a_2}{a_1 a_3}.$$

One obtains

$$c_1 = e^{\frac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3)} = \frac{a_1}{a_2 a_3}$$

by repeating the above argument with the isometry $M_{\mathcal{B}}$ induced by the matrix

$$\mathcal{B} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

□

Definition 2.5
A decorated ideal quadrilateral is defined analogously to a decorated ideal triangle.

A decorated ideal quadrilateral can be decomposed into two decorated ideal triangles in two ways:

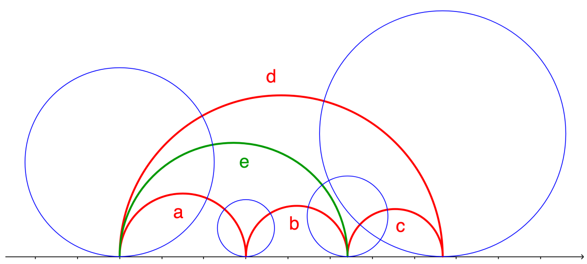


Figure 2: Triangulated decorated ideal quadrilateral

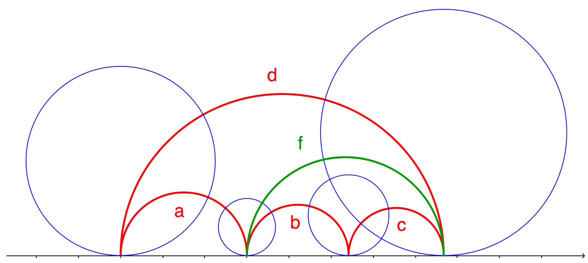


Figure 3: Triangulated decorated ideal quadrilateral

Lemma 2.4 (Ptolemy relation)
The six weights a, b, c, d, e, f are related by the Ptolemy relation

$$ef = ac + bd.$$

Proof. By Lemma 2.3 it holds:

$$\frac{e}{cd} = \frac{a}{df} + \frac{b}{cf}$$

which implies the result. □

3 Modular Torus

Definition 3.1
The modular group is the subgroup of $Isom^+(\mathbb{H})$ *consisting of all elements of the form*

$$M(z) = \frac{az+b}{cz+d}, \text{ where } a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1.$$

Definition 3.2
Let $\mathcal{G}$ *be the subgroup of the modular group generated by* $\alpha(z) = \frac{z-1}{-z+2}$ *and* $\beta(z) = \frac{z+1}{z+2}$ *and let* $\sim$ *be the equivalence relation on* H^2 *defined by*

$$x \sim y \iff \exists g \in \mathcal{G} \text{ such that } g(x) = y,$$

i.e. $x \sim y \iff x$ *and* y *are on the same orbit of the group action*

$$\phi : \mathcal{G} \times H^2 \to H^2, \quad (\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x) \mapsto \frac{ax+b}{cx+d}.$$

The quotient space $M = H^2/\sim$ *is called the modular torus.*

Next, we collect a few facts about the modular group, the group $\mathcal{G}$ and the isometries of the modular torus. More detail and some proofs can be found in [Kon].

Lemma 3.1 *1. The modular group is generated by the elements*

$$-\frac{1}{z} \text{ and } z+1.$$

2. $\mathcal{G}$ *is the commutator subgroup of the modular group.*

3. $\mathcal{G}$ *is a normal subgroup of the modular group with index six. The quotient group is the group of orientation preserving isometries of the modular torus.*

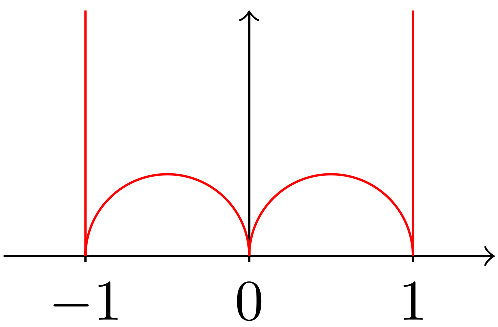
4. *The group of isometries of the modular torus is the quotient group of the subgroup of* $Isom(\mathbb{H})$ *consisting of all elements of the form*

$$M(z) = \frac{az+b}{cz+d}, \text{ where } a,b,c,d \in \mathbb{Z} \text{ and } |ad-bc| = 1$$

modulo $\mathcal{G}$*, has 12 elements and is generated by the equivalence classes of the elements*

$$-\frac{1}{z}, \; z+1 \text{ and } -\bar{z}.$$

Theorem 3.1
A fundamental domain of the modular torus is given by



$$F := \{z \in H^2 : -1 \le Re(z) \le 1\} \setminus \{z \in H^2 : |z + \frac{1}{2}| \le \frac{1}{2} \text{ or } |z - \frac{1}{2}| \le \frac{1}{2}\}.$$

Proof. The result is a consequence of Lemma 3.1 and Lemma 5 of [HT19]. □

Remark 1. *It holds:*

$$\alpha(1) = 0, \quad \beta(-1) = 0, \quad \alpha(\infty) = -1 \text{ and } \beta(\infty) = 1$$

Therefore, the modular torus is a torus with one point $(-1 \sim 0 \sim 1 \sim \infty)$ *removed, i.e. a once punctured hyperbolic torus.*

2. *By decomposing the ideal quadrilateral with vertices* -1, 0, 1 *and* ∞ *into two ideal triangles we obtain an ideal triangulation of the modular torus.*

Definition 3.3
The modular torus together with a choice of horocycle at the cusp is called decorated modular torus.

The decorated modular torus can be viewed as two congruent decorated ideal triangles that are glued together along their edges in a way that the horocycles fit together. In the following, we denote the weights of these two ideal decorated triangles by (a, b, c).

Example (Decorated Modular Tori)

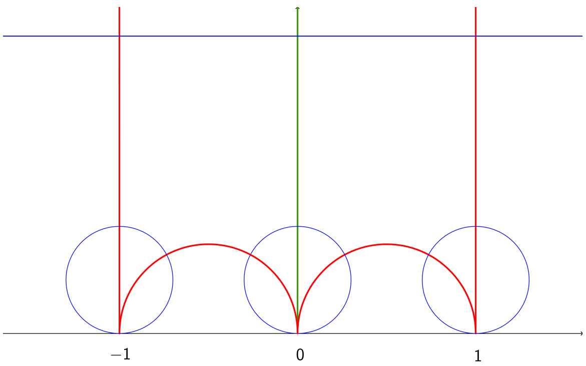


Figure 4: Decorated modular torus with ideal triangulation

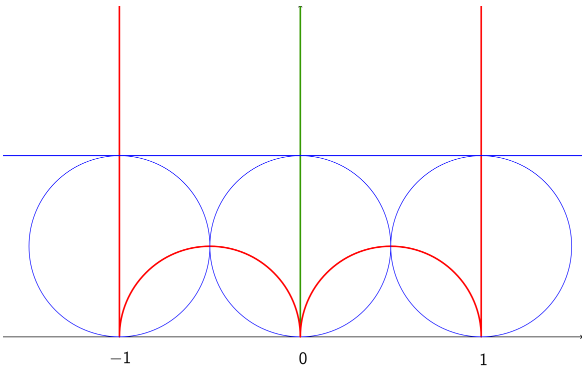


Figure 5: Decorated modular torus with ideal triangulation

Lemma 3.2
The total length of the horocycle of a decorated modular torus is given by

$$l = 2\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right).$$

Proof. Due to Lemma 2.3, the total length of the horocyclic arcs of a decorated ideal triangle with weights (a, b, c) is given by

$$\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}.$$

Hence,

$$l = 2\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right).$$

□

Remark
If $l = 6$*, then the weights* (a, b, c) *satisfy Markov's equation. Therefore, we consider from now on the decorated modular torus whose horocycle has total length 6. One obtains it by gluing two decorated ideal triangles with weights* $(1, 1, 1)$*.*

Lemma 3.3
If the triangulation with weights $(1, 1, 1)$ *and the decoration of the modular torus together with the horocycle of length 6 is lifted to the hyperbolic plane, then one obtains the Farey tessellation with Ford circles.*

Proof. We refer to section 3 of [Pfe15]. □

This means that the collection of images of the fundamental domain of the decorated modular torus with ideal triangulation as in Figure 5 under the isometries of the hyperbolic plane that are contained in $\mathcal{G}$, is the Farey tessellation with Ford circles.

Remark
Recall from [Bol20] the definition of Markov triples and neighbouring Markov triples. There are three involutions σ_k on the set of Markov triples that map any Markov triple (a, b, c) to its neighbours:

- $\sigma_1(a, b, c) = (\frac{b^2+c^2}{a}, b, c)$
- $\sigma_2(a, b, c) = (a, \frac{a^2+c^2}{b}, c)$
- $\sigma_3(a, b, c) = (a, b, \frac{a^2+b^2}{c})$

The following proposition is the main result of this section and establishes a connection between Markov triples and ideal triangulations of the decorated modular torus.

Proposition 3.1 (Markov triples and ideal triangulations) *1. A triple $\tau = (a, b, c)$ of positive integers is a Markov triple if and only if there is an ideal triangulation of the decorated modular torus whose three edges have weights a, b and c. This triangulation is unique up to the 12-fold symmetry of the modular torus.*

2. If T is an ideal triangulation of the decorated modular torus with edge weights $\tau = (a, b, c)$, and if T' is an ideal triangulation obtained from T by performing a single edge flip, then the edge weights of T' are $\tau' = \sigma_k\tau$, with $k \in \{1, 2, 3\}$ depending on which edge was flipped.

Proof. We only present one idea of the proof of the statement. More detail can be found in section 12 of [Spr17].

Let (a, b, c) be a Markov triple.

To obtain an ideal triangulation of the decorated modular torus with weights (a, b, c), one has to follow the Markov tree leading from $(1, 1, 1)$ to (a, b, c) and perform the corresponding edge flips on the projected Farey tessellation.

As an example, we execute three edge flips to get an ideal triangulation of the decorated modular torus with weights $(2, 5, 29)$.

The edge a forms a diagonal of the ideal quadrilateral with vertices -1, 0, 1 and ∞ which is a fundamental domain of the modular torus by Theorem 3.1.

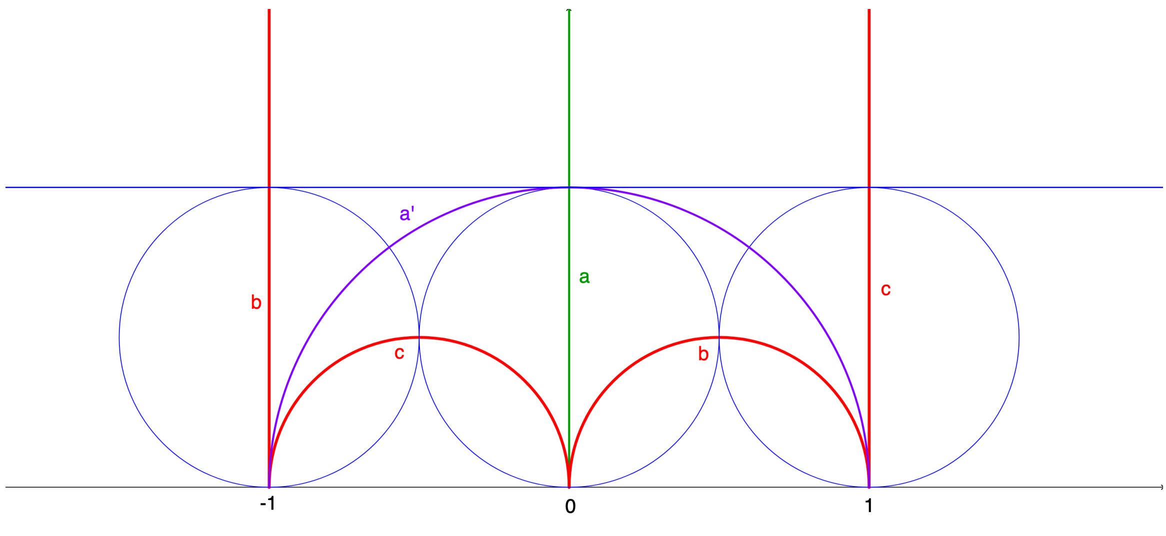


Figure 6: Flip of edge a

We get by the Ptolemy relation:

$$(a', b, c) = (\frac{b^2 + c^2}{a}, b, c) = \sigma_1(a, b, c).$$

Since $a = b = c = 1$, we get an ideal triangulation of the decorated modular torus with weights $(2, 1, 1)$.

Next, we want to flip edge b. Therefore, we look for an ideal quadrilateral that is a fundamental domain of the modular torus and has the edge b as a diagonal.

The isometry of the hyperbolic plane

$$\beta^{-1}(z) = \frac{2z-1}{-z+1}$$

sends the ideal quadrilateral with vertices -1, 0, 1 and ∞ to the ideal quadrilateral with vertices $-\frac{3}{2}$, -1, ∞ and -2. Hence, the ideal quadrilateral with vertices $-\frac{3}{2}$, -1, 1 and ∞ is a fundamental domain of the modular torus.

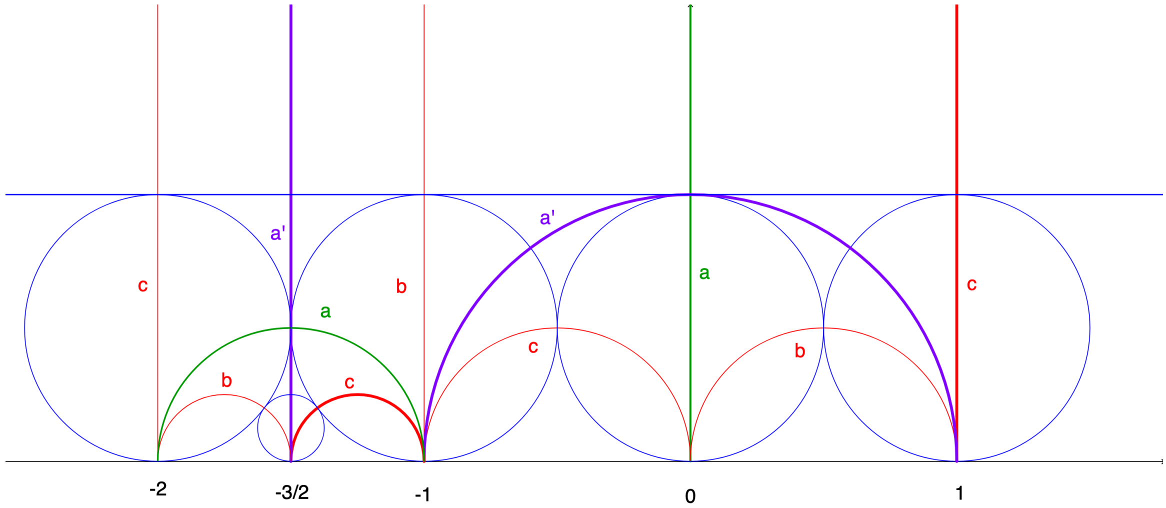


Figure 7: Looking for a new ideal quadrilateral

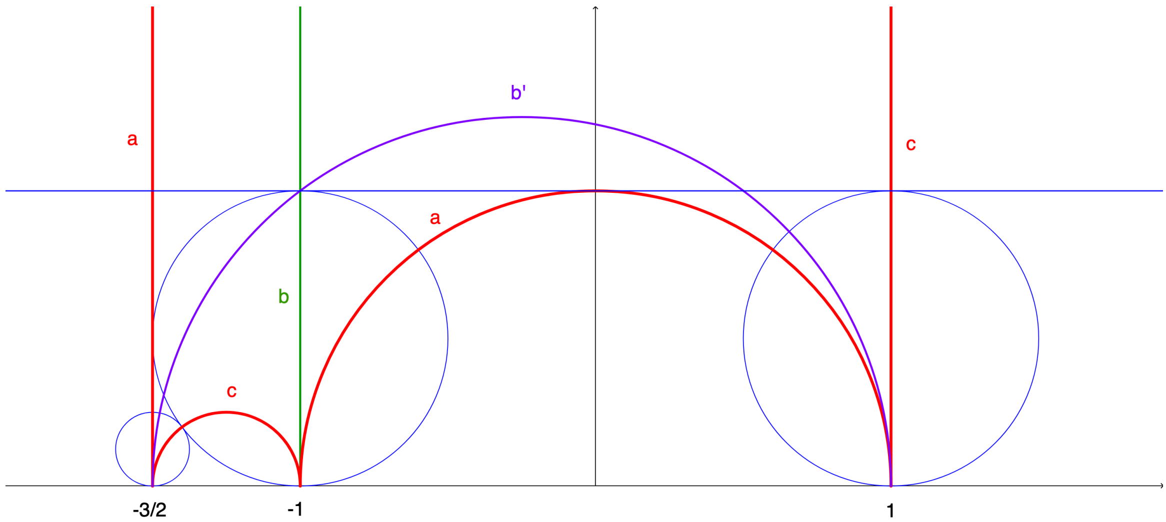


Figure 8: Flip of edge b

We get by the Ptolemy relation:

$$(a, b', c) = (a, \frac{a^2+c^2}{b}, c) = \sigma_2(a, b, c).$$

Since $a = 2$ and $b = c = 1$, we get an ideal triangulation of the decorated modular torus with weights $(2, 5, 1)$.

Finally, we want to flip edge c. In order to do that, we proceed as before.

The isometry of the hyperbolic plane

$$(\beta^{-1} \circ \alpha)(z) = \frac{3z-4}{-2z+3}$$

sends the ideal quadrilateral with vertices $-\frac{3}{2}$, -1, 1 and ∞ to the ideal quadrilateral with vertices $-\frac{17}{12}$, $-\frac{7}{5}$, -1 and $-\frac{3}{2}$. Hence, the ideal quadrilateral with vertices $-\frac{3}{2}$, $-\frac{17}{12}$, -1 and 1 is a fundamental domain of the modular torus.

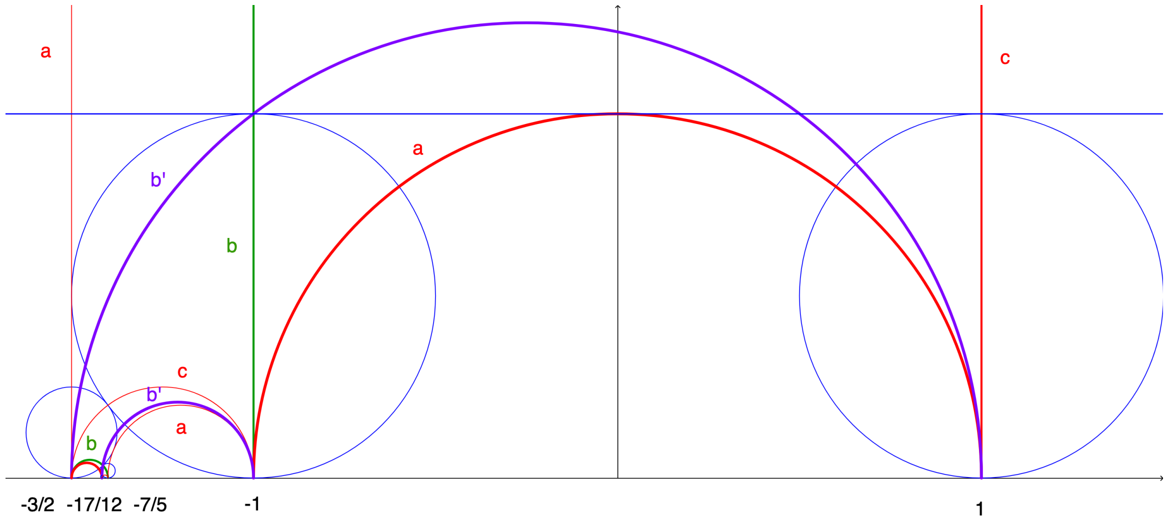


Figure 9: Looking for a new decorated ideal quadrilateral

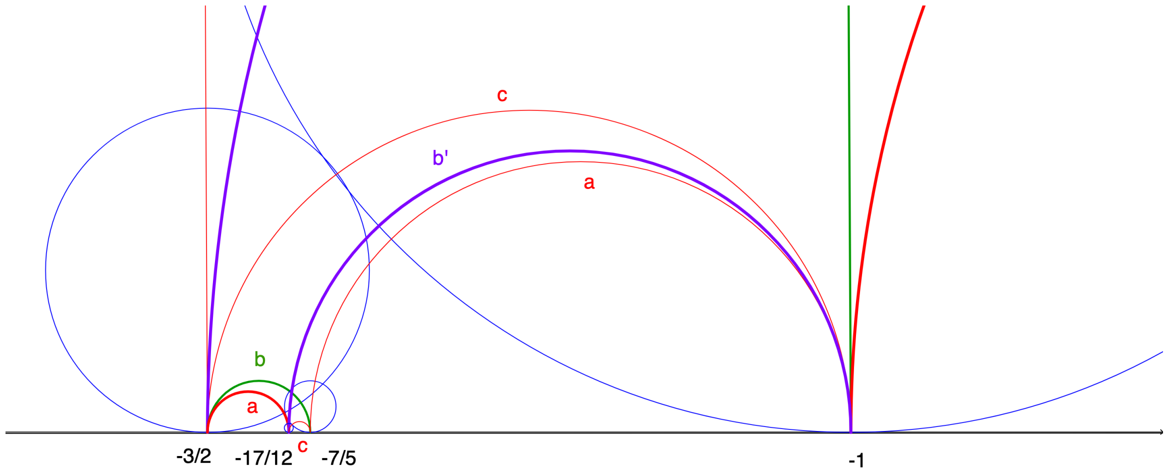


Figure 10: Magnification of the previous figure

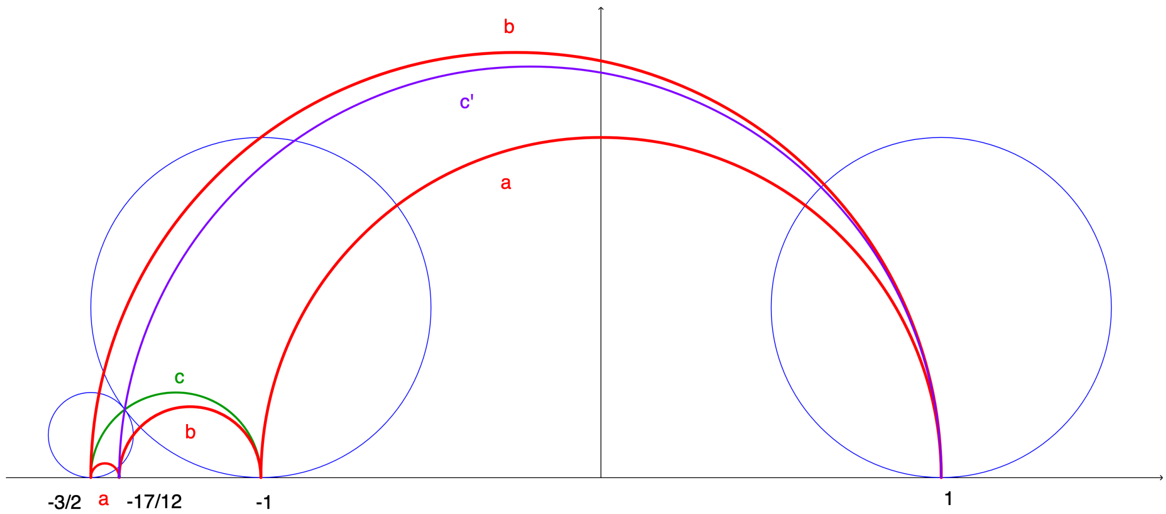


Figure 11: Flip of edge c

We get by the Ptolemy relation:

$$(a, b, c') = (a, b, \frac{a^2 + b^2}{c}) = \sigma_3(a, b, c).$$

Since $a = 2$, $b = 5$ and $c = 1$, we get an ideal triangulation of the decorated modular torus with weights $(2, 5, 29)$.

As a final remark concerning this proof, we examine whether it makes a difference to perform the flip at edge b, which we did above, at the geodesic connecting -1 and ∞ or at the geodesic connecting 0 and 1.

The isometry of the hyperbolic plane

$$\beta(z) = \frac{z+1}{z+2}$$

sends the ideal quadrilateral with vertices −1, 0, 1 and ∞ to the ideal quadrilateral with vertices 0, $\frac{1}{2}$, $\frac{2}{3}$ and 1. Hence, the ideal quadrilateral with vertices −1, 0, $\frac{2}{3}$ and 1 is a fundamental domain of the modular torus.

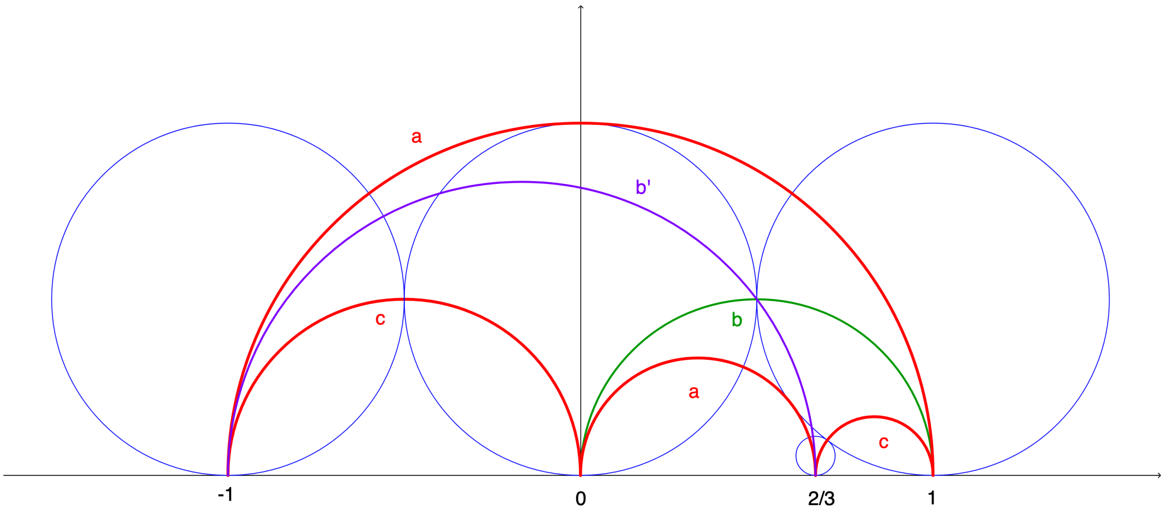


Figure 12: Flip of edge b (second version)

As above, we get an ideal triangulation of the decorated modular torus with weights $(2, 5, 1)$.

The isometry of the modular torus that is induced by the isometry of the hyperbolic plane

$$M(z) = \frac{-1}{z}$$

sends the ideal quadrilateral with vertices $-\frac{3}{2}$, −1, 1 and ∞ to the ideal vertices quadrilateral with vertices $\frac{2}{3}$, 1, −1 and 0.

Hence, one does not necessarily get the same ideal triangulation of the modular torus if one performs a flip at different representatives of an edge, but the two ideal triangulations are related by an isometry of the modular torus. □

4 Geodesics crossing a decorated ideal triangle

In this section we consider the following geometric optimization problem:

Given a decorated ideal triangle, find among all geodesics intersecting the sides a_1 and a_2, a geodesic that maximizes the minimum of the signed distances to the three horocycles at the vertices.

Definition 4.1 *1. A geodesic bisects a side of a decorated ideal triangle if it intersects the side in the point at equal distance to the two horocycles at the ends of the side.*

2. The perpendicular bisector of a side of a decorated ideal triangle is the geodesic that intersects the side in the point at equal distance to the two horocycles at the ends of the side at right angles.

Lemma 4.1
A geodesic that bisects two sides of a decorated ideal triangle has equal signed distance to all three horocycles of the decorated ideal triangle.

Proof. By Lemma 2.2, it suffices to consider the ideal triangle with vertices $v_1 = 0$, $v_2 = 1$ and $v_3 = \infty$ and horocycles $h_1 = h(0, A)$, $h_2 = h(B, B)$ and $h_3 = h(C, 0)$, where $A, B, C \in \mathbb{R}_{>0}$, and the geodesic g that bisects the sides a_1 and a_2.

Let $Q_1 = 1 + \gamma_1 i$ be the point where g and a_1 intersect and let $Q_2 = \gamma_2 i$ be the point where g and a_2 intersect.

It holds:

$$\log \frac{C^2}{\gamma_2} = \log A^2\gamma_2 \text{ and } \log \frac{C^2}{\gamma_1} = \log B^2\gamma_1$$

Therefore:

$$A^2\gamma_2^2 = C^2 = B^2\gamma_1^2 \tag{2}$$

Furthermore, Q_1 and Q_2 lie on the euclidean half circle with center

$$m = \frac{1}{2}(\gamma_1^2 - \gamma_2^2 + 1)$$

and radius

$$r = \sqrt{\frac{1}{4}(\gamma_1^2 - \gamma_2^2)^2 + \frac{1}{2}(\gamma_1^2 + \gamma_2^2) + \frac{1}{4}}.$$

This implies that

$$\gamma_1^2 + (m-1)^2 = r^2 = \gamma_2^2 + m^2. \tag{3}$$

We define $f(x, y) = x^2 - 2mxy + (m^2 - r^2)y^2$.

It holds:

- $\operatorname{disc}(f) = 4r^2$
- $f(0, A) = A^2(m^2 - r^2) = A^2\gamma_2^2$
- $f(B, B) = B^2 - 2mB^2 + (m^2 - r^2)B^2 = -(r^2 - (1-m)^2)B^2 = -\gamma_1^2 B^2$

Therefore:

$$\begin{aligned} d(h_1, g) &= d(h(0, A), g(f)) \\ &= \log \frac{A^2\gamma_2^2}{r} \\ &= \log \frac{C^2}{r} + \log \frac{A^2\gamma_2^2}{C^2} \\ &= \log \frac{C^2}{r} + \log A^2\gamma_2 - \log \frac{C^2}{\gamma_2} \\ &= \log \frac{C^2}{r} = d(h_3, g) \end{aligned}$$

$$\begin{aligned} d(h_2, g) &= d(h(B, B), g(f)) \\ &= \log \frac{B^2\gamma_1^2}{r} \\ &= \log \frac{C^2}{r} + \log \frac{B^2\gamma_1^2}{C^2} \\ &= \log \frac{C^2}{r} + \log B^2\gamma_1 - \log \frac{C^2}{\gamma_1} \\ &= \log \frac{C^2}{r} = d(h_3, g) \end{aligned}$$

□

Corollary 4.1
A geodesic that bisects two sides of a decorated ideal triangle that is part of the Farey tessellation together with the Ford circles has signed distance

$$\log \frac{2}{\sqrt{5}}$$

to all three horocycles.

Proof. This is a consequence of Corollary 2.3 and the proof of Lemma 4.1. □

Proposition 4.1
We consider a decorated ideal triangle with weights $(a_1, a_2, a_3) \in \mathbb{R}^3$.

1. *If*
$$a_1^2 \leq a_2^2 + a_3^2 \text{ and } a_2^2 \leq a_1^2 + a_3^2, \tag{4}$$
then the geodesic g bisecting the sides a_1 and a_2 is the unique solution of the above optimization problem.

2. *If, for $(j, k) \in \{(1, 2), (2, 1)\}$,*
$$a_j^2 \geq a_k^2 + a_3^2, \tag{5}$$
then the perpendicular bisector $\tilde{g}$ of the side a_k is the unique solution of the above optimization problem. In this case, the minimal distance is attained for h_j and h_3,
$$d(h_j, \tilde{g}) = d(h_3, \tilde{g}) = \frac{\alpha_k}{2} \leq d(h_k, \tilde{g}).$$

Proof. By Lemma 2.2, it suffices to consider the ideal triangle with vertices $v_1 = 0$, $v_2 = 1$ and $v_3 = \infty$ and horocycles $h_1 = h(0, A)$, $h_2 = h(B, B)$ and $h_3 = h(C, 0)$, where $A, B, C \in \mathbb{R}_{>0}$.

For $j \in \{1, 2, 3\}$ we define P_j to be the point on g that is closest to h_j and denote the geodesic connecting P_j and v_j with g_j. If g_j is a euclidean half circle with center on the real axis, we denote its center by m_j and its radius by r_j. If g_j is a euclidean vertical line we define m_j to be the point where g_j hits the real axis and set $r_j = 0$.

It holds:

- $r_1 = |m_1|$ and $(m - m_1)^2 = r_1^2 + r^2$ because g and g_1 intersect at a right angle at P_1.
- $r_2 = |m_2 - 1|$ and $(m - m_2)^2 = r_2^2 + r^2$ because g and g_2 intersect at a right angle at P_2.
- $r_3 = 0$ and $m_3 = m$.

Therefore:

$$m_1 = \frac{m^2 - r^2}{2m} \quad \text{and} \quad m_2 = \frac{m^2 - r^2 - 1}{2(m-1)} \tag{6}$$

We distinguish between the following four cases:

1. P_3 lies strictly between P_1 and P_2:

   In this case, g is the unique solution of the optimization problem. By Lemma 4.1, g has equal distance to all three horocycles. In addition, any other geodesic crossing the sides a_1 and a_2 also crosses the ray from P_j to v_j for at least one $j \in \{1, 2, 3\}$. Therefore, it is closer to at least one of the horocycles.

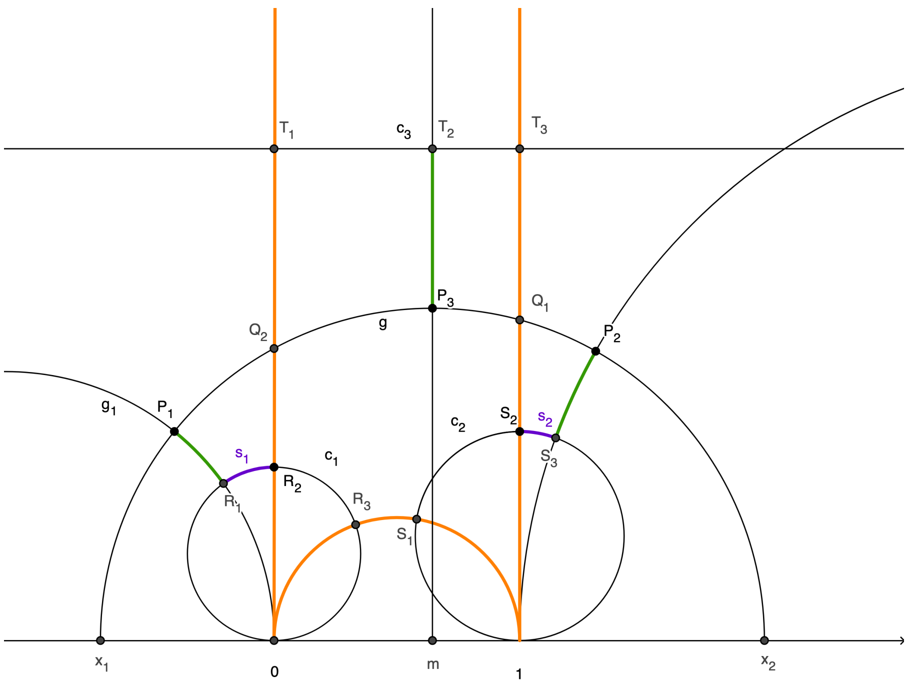


Figure 13: P_3 lies strictly between P_1 and P_2

2. P_1 lies strictly between P_3 and P_2:

   In this case, the perpendicular bisector of the side a_2 is the unique solution of the optimization problem. We denote it by b_2. The signed distance of b_2 to the horocycles h_1 and h_3 is half the truncated length of side a_2. The signed distance of b_2 and the h_2 is larger. Any other geodesic crossing a_2 is either closer to h_1 or to h_3.

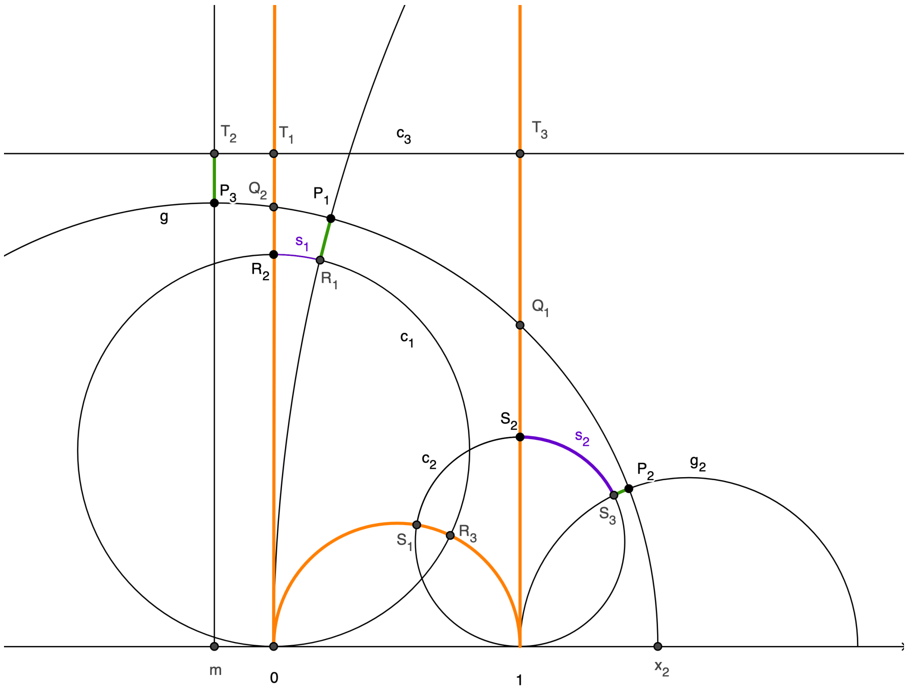


Figure 14: P_1 lies strictly between P_3 and P_2

3. P_2 lies strictly between P_1 and P_3:

   In this case, the perpendicular bisector of the side a_1 is the unique solution of the optimization problem. We use the same arguments as in the second case.

4. $P_3 = P_1$ or $P_3 = P_2$:

   If $P_3 = P_1$, then g is simultaneously the geodesic connecting Q_1 and Q_2 and the perpendicular bisector of the side a_2.

   If $P_3 = P_2$, then g is simultaneously the geodesic connecting Q_1 and Q_2 and the perpendicular bisector of the side a_1.

It remains to show that the order of the points P_j on g depends on whether the inequalities (4) or (5) are satisfied.
We define

- s_1 to be the horocyclic arc of h_1 between R_1 and R_2 and
- s_2 to be the horocyclic arc of h_2 between S_2 and S_3.
- t_1 to be the horocyclic arc of h_3 between T_1 and T_2,
- t_2 to be the horocyclic arc of h_3 between T_2 and T_3,
- u_1 to be the horocyclic arc of h_1 between R_1 and R_3 and
- u_2 to be the horocyclic arc of h_2 between S_1 and S_3.

We denote the absolute value of the hyperbolic length of the horocyclic arcs we just defined by | horocyclic arc |.

claim: $|s_1| = |t_1|$, $|s_2| = |t_2|$ and $|u_1| = |u_2|$

$|t_1|$ and $|t_2|$ can be determined as follows:

$$|t_1| = \left| \int_0^m \frac{1}{C^2} dt \right| = \frac{|m|}{C^2} \quad \text{and} \quad |t_2| = \left| \int_m^1 \frac{1}{C^2} dt \right| = \frac{|1-m|}{C^2}$$

The isometry $\frac{1}{z}$ maps

- the horocycle h_1 to the horocycle $h(A, 0)$,

- the geodesic g_1 to the geodesic connecting $\frac{1}{2m_1}$ and ∞ and
- the side a_3 to the geodesic connecting 1 and ∞.

Therefore, R_1 is sent to $\frac{1}{2m_1} + A^2 i$, R_2 is sent to $A^2 i$ and R_3 is sent to $1 + A^2 i$.

It follows by (2),(3) and (6):

$$|s_1| = \left| \int_0^{\frac{1}{2m_1}} \frac{1}{A^2} dt \right| = \frac{1}{2A^2|m_1|} \quad \text{and} \quad |u_1| = \left| \int_{\frac{1}{2m_1}}^1 \frac{1}{A^2} dt \right| = \frac{|2m_1 - 1|}{2A^2|m_1|}$$
$$\implies |s_1| = \frac{|m|}{A^2|m^2 - r^2|} = \frac{|m|}{A^2\gamma_2^2} = \frac{|m|}{C^2} = |t_1|$$

The isometry $\frac{1}{\bar{z}-1}$ maps

- the horocycle h_2 to the horocycle $h(B, 0)$,
- the geodesic g_2 to the geodesic connecting $\frac{1}{2m_2-2}$ and ∞ and
- the side a_3 to the geodesic connecting -1 and ∞.

Therefore, S_1 is sent to $-1 + B^2 i$, S_2 is sent to $B^2 i$ and S_3 is sent to $\frac{1}{2m_2-2} + B^2 i$.

It follows by (2),(3) and (6):

$$|s_2| = \left| \int_0^{\frac{1}{2m_2-2}} \frac{1}{B^2} dt \right| = \frac{1}{2B^2|m_2 - 1|} \quad \text{and} \quad |u_2| = \left| \int_{-1}^{\frac{1}{2m_2-2}} \frac{1}{B^2} dt \right| = \frac{|2m_2 - 1|}{2B^2|m_2 - 1|}$$
$$\implies |s_2| = \frac{|m-1|}{B^2|(m-1)^2 - r^2|} = \frac{|m-1|}{B^2\gamma_1^2} = \frac{|m-1|}{C^2} = |t_2|$$

and

$$|u_1| = \frac{|r^2 - m^2 + m|}{A^2|r^2 - m^2|} = \frac{|\gamma_2^2 + m|}{A^2\gamma_2^2} = \frac{|\gamma_1^2 + (m-1)^2 - m^2 + m|}{B^2\gamma_1^2}$$
$$= \frac{|\gamma_1^2 + 1 - m|}{B^2\gamma_1^2} = \frac{|r^2 - m^2 + m|}{B^2|r^2 - (m-1)^2|} = |u_2|.$$

If P_3 lies strictly between P_1 and P_2, then it hods:

1. $c_1 = |u_1| - |s_1|$
2. $c_2 = |u_2| - |s_2|$
3. $c_3 = |t_1| + |t_2|$

This implies that

$$2|t_1| = -c_1 + c_2 + c_3 = -\frac{a_1}{a_2a_3} + \frac{a_2}{a_1a_3} + \frac{a_3}{a_1a_2} = \frac{-a_1^2 + a_2^2 + a_3^2}{a_1a_2a_3}$$

and

$$2|t_2| = c_1 - c_2 + c_3 = \frac{a_1}{a_2a_3} - \frac{a_2}{a_1a_3} + \frac{a_3}{a_1a_2} = \frac{a_1^2 - a_2^2 + a_3^2}{a_1a_2a_3}.$$

P_3 lies strictly between P_1 and P_2 if and only if $|t_1| > 0$ and $|t_2| > 0$.

Therefore, P_3 lies strictly between P_1 and P_2 if and only if the inequalities $a_2^2 + a_3^2 > a_1^2$ and $a_1^2 + a_3^2 > a_2^2$ are satisfied.

The other cases are treated similarly.

□

The last major result of this paper presents an application of Proposition 4.1 and relates hyperbolic geometry with the theory of indefinite binary quadratic forms.

For the proof of our last theorem we need the following lemma:

Lemma 4.2
If a geodesic $g = g(f)$, where f is an indefinite binary quadratic form, bisects two sides of a decorated ideal triangle that is part of the Farey tessellation together with the Ford circles, then f is equivalent to an indefinite binary quadratic form of the type

$$\alpha x^2 - \alpha xy - \alpha y^2, \text{ where } \alpha \in \mathbb{R} \setminus \{0\}.$$

Proof. By Corollary 2.3, there is an isometry M that maps g to $g(\tilde{f})$, where

$$\tilde{f}(x, y) = x^2 - xy - y^2$$

because $g(\tilde{f})$ bisects the sides a_1 and a_2 of the decorated ideal triangle with vertices 0, 1 and ∞ and horocycles $h_1 = h(0, 1)$, $h_2 = h(1, 1)$ and $h_3 = h(1, 0)$.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in GL_2(\mathbb{R})$$

be such that $|\det(A)| = 1$ and $M_A = M$.

Therefore,

$$M_A(g(f)) = g(\tilde{f}),$$

which implies by Lemma 1.2 that

$$g(f) = M_{A^{-1}}(g(\tilde{f})) = g(\tilde{f} \circ A).$$

Hence, we get by Lemma 1.1 that

$$f = \alpha(\tilde{f} \circ A) = (\alpha\tilde{f}) \circ A, \text{ for some } \alpha \in \mathbb{R} \setminus \{0\}.$$

□

Theorem 4.1 (Korkin and Zolotarev)
Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form with real coefficients.

1. *If f is equivalent to an indefinite binary quadratic form of the type*

$$\alpha x^2 - \alpha xy - \alpha y^2, \text{ where } \alpha \in \mathbb{R} \setminus \{0\},$$

   then

$$\inf_{(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{|f(p, q)|}{\sqrt{disc(f)}} = \frac{1}{\sqrt{5}}.$$

2. *Otherwise,*

$$\inf_{(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{|f(p, q)|}{\sqrt{disc(f)}} < \frac{1}{\sqrt{5}}.$$

Proof. 1. There is a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

such that $|\det(A)| = 1$ and

$$f(x, y) = \alpha(a_{11}x + a_{12}y)^2 - \alpha(a_{11}x + a_{12}y)(a_{21}x + a_{22}y) - \alpha(a_{21}x + a_{22}y)^2.$$

It holds:

$$\frac{|f(x, y)|}{\sqrt{\text{disc}(f)}} = \frac{|\alpha||(\tilde{f} \circ A)(x, y)|}{\sqrt{\alpha^2 \, \text{disc}(\tilde{f} \circ A)}} \text{ where } \tilde{f}(x, y) = x^2 - xy - y^2.$$

Hence, it suffices to prove the result for the case that f is equivalent to $x^2 - xy - y^2$.

We conclude the proof of the first part of the theorem by the fact that for every geodesic g there exists a Ford circle h such that g and h intersect, by Corollary 1.1 and by Proposition 1.1.

2. We choose an arbitrary decorated ideal triangle that is part of the Farey tessellation together with the Ford circles such that $g(f)$ intersects two of its sides and denote it by $\mathcal{T}$.
   The weights of $\mathcal{T}$ are given by $(1, 1, 1)$.

   By Lemma 4.2, f does not bisect two sides of $\mathcal{T}$. Hence, by the first part of Proposition 4.1, the minimum of the signed distances of $g(f)$ and the horocycles of $\mathcal{T}$ is strictly smaller than the minimum of the signed distances of the geodesic bisecting the sides a_1 and a_2 of $\mathcal{T}$ and the horocycles of $\mathcal{T}$.

   Therefore, Corollary 4.1 implies that the minimum of the signed distances of $g(f)$ and the horocycles of $\mathcal{T}$ is strictly less than $\log \frac{2}{\sqrt{5}}$.

   This implies by Proposition 1.1 that

$$\exists\,(p,q) \in \mathbb{Z}^2 \setminus \{0\} \text{ such that } d(h(p,q), g(f)) = \log \frac{2|f(p,q)|}{\sqrt{\operatorname{disc}(f)}} < \log \frac{2}{\sqrt{5}}.$$

   Finally, we conclude that

$$\inf_{(p,q)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{|f(p,q)|}{\sqrt{\operatorname{disc}(f)}} < \frac{1}{\sqrt{5}}.$$

□

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