# 5. Size of the set of badly approximable numbers

Recall:

The set of badly approximable number is defined by

$$\mathrm{Bad} := \{x \in \mathbb{R} : \inf_{q \in \mathbb{N}} q\|qx\| > 0\}$$

$$= \{x \in \mathbb{R} : \gamma(x) := \liminf_{q \to \infty} q\|qx\| > 0 .$$

Hurwitz proved that for every $x \in \mathbb{R}$ we have $q\|qx\| < 1/\sqrt{5}$ for infinitely many $q > 0$.

Hence, by Hurwitz we get that

$$0 \leq \gamma(x) \leq \frac{1}{\sqrt{5}} .$$

Furthermore we know that the golden ratio is badly approximable and in particular $\mathrm{Bad} \neq \emptyset$.

Goal of today's seminar

is to „measure the size" of Bad.

The main goal will be:

a) $m(\mathrm{Bad}) = 0$

b) $\dim \mathrm{Bad} = 1$

where $m$ is the Lebesgue-measure and $\dim$ denotes the Hausdorff dimension

Definition:

For a function $\psi : \mathbb{N} \to \mathbb{R}^+$, a real number $x$ is said to be $\psi$-approximable if there are

infinitely many $q \in \mathbb{N}$ such that
$\|qx\| < \psi(q)$.

Since the set of $\psi$-approximable numbers is invariant under translations by integer vectors, we restrict in this section to $\psi$-approximable numbers in the unit intervall $I = [0,1)$. We denote it by

$$W(\psi) := \{x \in [0,1] : \|qx\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

The existence of badly approximable numbers implies that there exist approximating functions $\psi$ for which $W(\psi) \neq I$. The fact that $m(\mathrm{Bad}) = 0$ implies that we can have $W(\psi) \neq I$ while $m(W(\psi)) = 1$.

Example: By Hurwitz's Theorem we know:
$W(\psi) = I$ if $\psi(q) = 1/(\sqrt{5}q)$ and
$W(\psi) \neq I$ if $\psi(q) = 1/((\sqrt{5}+\varepsilon)q)$ for all $\varepsilon > 0$.

Let $\psi_c : \mathbb{N} \to \mathbb{R}^+$ be

$$\psi_c(q) = c/q \quad \text{for } c > 0.$$

By definition of the badly approximable numbers we get

$$\mathrm{Bad} = \bigcup_{c>0} (\mathbb{R} \setminus W(\psi_c)) = \bigcup_{m=1}^{\infty} (\mathbb{R} \setminus W(\psi_{1/m})).$$

Khintchine's Theorem

For proving that $m(\mathrm{Bad}) = 0$, we are going to prove a more general theorem

Theorem (Khintchine)

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a monotonic function. Then

$$m(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty, \\ 1 & \text{if } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

Since the proof of the divergent case is rather long, we only prove the case, where

1) $\quad \sum_{q=1}^{\infty} \psi(q) < \infty.$

Proof (of case 1))

The main idea of the proof is to observe that we can write $W(\psi)$ as a limsup of sets.

Define

$$E_q(\psi) = \{x \in I : \|qx\| < \psi(q)\}.$$

The set $W(\psi)$ is the set of real numbers which lie in infitely many sets $E_q(\psi)$ with $q = 1, 2, \dots$ that is,

$$W(\psi) = \limsup_{q \to \infty} E_q(\psi) := \bigcap_{Q=1} \bigcup_{q=Q}^{\infty} E_q(\psi).$$

Hence for every $Q > 1$

$$W(\psi) \subset \bigcup_{q=Q}^{\infty} E_q(\psi).$$

Since a measure is non-negative, monotonic and subadditive we have

2) $\quad 0 \leq m(W(\psi)) \leq m\left(\bigcup_{q=Q}^{\infty} E_q(\psi)\right) \leq \sum_{q=Q}^{\infty} m(E_q(\psi)).$

By 1) we have that $\psi(q) \to 0$ as $q \to \infty$. Hence there exists $Q_0$ such that for any $q \geq Q_0$ we have that $\psi(q) < \frac{1}{2}$. In this case we can write

$$E_q(\psi) = \bigcup_{p=0}^{q} \{x \in [0,1] : |qx - p| < \psi(q)\}.$$

This is the union of $q-1$ disjoint intervals of length $2\psi(q)/q$ plus two intervals of length $\psi(q)/q$. Therefore, since $m$ is $\sigma$-additive, we have that

$$m(E_q(\psi)) = m\left(\bigcup_{p=0}^{q} \{x \in [0,1] : |qx - p| < \psi(q)\}\right) =$$

$$= \sum_{p=0}^{q} m(\{x \in [0,1] : |qx - p| < \psi(q)\}) = 2\psi(q).$$

Plugging this into (3.6) gives

$$0 \leq m(W(\psi)) \leq \sum_{q=Q}^{\infty} 2\psi(q).$$

Since this holds for every $Q$, we get as $Q \to \infty$
$m(W(\psi)) = 0$ as required. $\square$

Remark

The assumption that $\psi$ is monotonic is only required in the divergent case (in the proof above we never used monotonicity of $\psi$).

Note that

$$m(W(\psi)) = 1 \quad \text{if } \psi(q) = 1/q \log q.$$

Corollary

Let Bad be the set of badly approximable numbers. Then

$$m(\mathrm{Bad}) = 0.$$

Proof:

Let $\psi(q) = 1/(q \log q)$ and $\psi_m = 1/(mq)$.
Note that $\psi(q) < \psi_m(q)$ for a fixed $m$ and $q$ large enough.
We have

$$\mathrm{Bad} \cap I = \bigcup_{m=1}^{\infty} (I \setminus W(\psi_m)) \subseteq I \setminus W(\psi),$$

since

$$W(\psi_m) = \{x \in I : \|qx\| < \frac{1}{mq} \text{ for infinitely many } q \in \mathbb{N}\}$$
$$\supseteq W(\psi).$$

By Khintchine's Theorem, $m(W(\psi))=1$.
Thus $m(I\setminus W(\psi))=0$ and so $m(\mathrm{Bad}\cap I)=0$. $\square$

## Hausdorff measure and dimension

**Definition**
Let $X\subset\mathbb{R}^n$ and $\rho>0$. Any finite or countable collection $\{B_i\}$ of balls of diameter $d_i<\rho$ such that

$$X\subset\bigcup_i B_i$$

will be called a $\rho$-cover of $X$.

For $s\geq 0$ and $\rho>0$ define

$$\mathcal{H}^s_\rho(X)=\inf\left\{\sum_i d_i^s : \{B_i\} \text{ is a } \rho\text{-cover of } X\right\},$$

where the infimum is taken over all possible $\rho$-covers of $X$.
Observe that as $\rho$ decreases $\mathcal{H}^s_\rho$ increases, since for $\rho'<\rho$ every $\rho'$-cover of $X$ is also a $\rho$-cover.
Therefore, the following (finite or infinite) limit exists

$$\mathcal{H}^s(X)=\lim_{\rho\to 0+}\mathcal{H}^s_\rho(X)$$

and is called the $s$-dimensional Hausdorff measure of $X$.

**Properties**

(i) If $s=n\in\mathbb{N}$, then $\mathcal{H}^s(X)=c_n m_n(X)$ for some constant $c_n$, where $m_n$ denotes the $n$-dimensional Lebesgue-measure. Note $c_1=1$.
(ii) If $t>s$ and $\mathcal{H}^s(X)<\infty$ then $\mathcal{H}^t(X)=0$

Proof: We have by the above observation $\mathcal{H}^s_\rho\leq\mathcal{H}^s(X)<\infty$

for every $\rho>0$. Hence for $\rho>0$ there is a $\rho$-cover $\{B_i\}$ of $X$ of balls of diameters $d_i$ such that

$$\sum_i d_i^s < \mathcal{H}^s(X)+1<\infty.$$

Then $$\sum_i d_i^t = \sum_i d_i^s \times d_i^{t-s} \leq \sum_i d_i^s \times \rho^{t-s} \leq (\mathcal{H}^s(X)+1)\times \rho^{t-s}.$$

Therefore,

$$\mathcal{H}^t_\rho(X) \leq (\mathcal{H}^s(X)+1)\times \rho^{t-s} \to 0 \quad \text{as } \rho\to 0+ .$$

Thus $\mathcal{H}^t(X)=0$.

(iii) For any given set $X\subset\mathbb{R}^n$, $\mathcal{H}^s(X)=0$ if $s>n$.

(iv) For any given set $X\subset\mathbb{R}^n$, there is a unique number $s_0\geq 0$ such that for every $s\geq 0$ we have that

$$\mathcal{H}^s(X)=\begin{cases} 0 & s>s_0, \\ \infty & s<s_0. \end{cases}$$

Note $\mathcal{H}^{s_0}(X)\in[0,\infty]$.

This unique number $s_0$ is called the Hausdorff dimension of $X$ and will be denoted by $\dim X$.

Examples:

1) The Hausdorff dimension of a point is $0$, since for every $s\geq 0$ the $s$-dimensional Hausdorff measure is $0$.

2) The Hausdorff dimension of a countable set is $0$, by 1) and the $\sigma$-additivity of measures.

3) The Hausdorff dimension of a non-empty interval is $1$.

4) The Hausdorff dimension of a non-empty square is $2$.

5) The Hausdorff dimension of $\mathbb{R}^n$ is $n$.

In example 1)–5) the Hausdorff dimension is equal to the topological dimension. This is not true in general as we will see a counterexample.

Observe by (i) we have that

$$\mathcal{H}^1(\text{Bad}) = m(\text{Bad}) = 0$$

and therefore by (iv) we get

$$\dim \text{Bad} \leq 1.$$

Theorem (Jarník)

$$\dim \text{Bad} = 1.$$

Note that this implies $\dim \text{Bad} = \dim [0,1]$, even though Bad is properly contained in $[0,1]$.

We will not prove the second inequality for Jarnik's theorem but show at the example of the middle third cantor set how one can derive upper and lower bounds for the Hausdorff dimension. We will also see that the Hausdorff dimension can be irrational.

The upper bounds can be obtained by constructing coverings.

Lemma

Let $X \subset \mathbb{R}^n$ and $s > 0$. Suppose there exists a constant $C > 0$ such that for any $\rho > 0$ there exists a $\rho$-cover $\{B_i\}$ of $X$ with

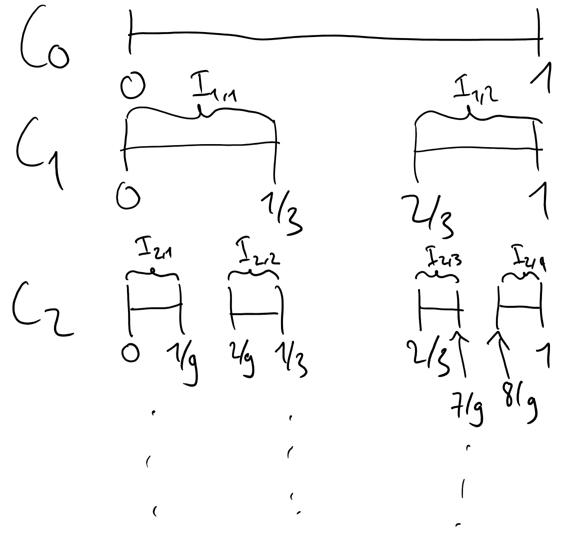
$$\sum_i \operatorname{diam}(B_i)^s < C,$$

then $\dim X \leq s$.

Proof:

Since the above inequality holds for any $\rho > 0$, we get $\mathcal{H}^s(X) \leq C$ and by property (ii) the lemma follows. $\square$

Denote by $C$ the middle third Cantor set, which is defined iteratively. Let $C_0 = [0,1]$ and $C_1$ be $C_0$ without the middle third interval. $C_1$ is then an union of 2 closed intervals $I_{1,1} = [0, 1/3]$ and $I_{1,2} = [2/3, 1]$. If we repeat the above procedure with $I_{1,1}$ and $I_{1,2}$, we get that $C_2$ is a union of 4 closed intervals. Finally, we get $C_n$ as a union of $2^n$ closed intervals of the length $3^{-n}$, which we denote by $I_{n,1}, \dots, I_{n,2^n}$.




The middle third cantor set is then defined by

$$C = \bigcap_{n \geq 0} C_n .$$

By definition we have that the collection of intervalls $\{I_{n,j}\}$ is a cover of $C$ for each $n$.

Moreover for $\rho > 0$ there exists a sufficent large $n$ such that $\{I_{n,j}\}$ is a $\rho$-cover of $C$. Hence, we get

$$\sum_j \operatorname{diam}(I_{n,j})^s = 2^n 3^{-ns} = 2^{n(1-s\,(\log 3/\log 2))}$$

which is equal to 1 by setting $s = \log 2/\log 3$.

We get by the lemma that

$$\dim C \le \frac{\log 2}{\log 3}.$$

To prove a lower bound is often more complicated, since we need to consider all the $\rho$-covers.

We want to show that

$$s := \frac{\log 2}{\log 3} \le \dim C,$$

which will then imply the equallity.

The idea is to use that $C$ is a closed and bounded subset as a intersection of closed set and therefore compact.

Therefore every $\{B_i\}$ $\rho$-cover of $C$ contains a finite subcover. Hence we can assume that $\{B_i\}$ is finite. For each $B_i$ there exists exactly one $k \in \mathbb{N}$ such that

$$3^{-(k+1)} \le \operatorname{diam}(B_i) < 3^{-k}.$$

Therefore $B_i$ intersects at most one interval of $C_k$.

If $j \ge k$ intersects at most

$$2^{j-k} = 2^j 3^{-sk} \le 2^j 3^s \operatorname{diam}(B_i)^s$$

intervals o $C_j$.

Since $\{B_i\}$ is finite we can find $j \in \mathbb{N}$ such that

$$3^{-(j+1)} \le \operatorname{diam} B_i$$

for all $B_i$. Since $\{B_i\}$ is a cover of $C$ it intersects

every intervall of $C_j$.
Putting everything together we get that

$$2^j \leq \sum_i 2^j 3^s \operatorname{diam}(B_i)^s,$$

hence

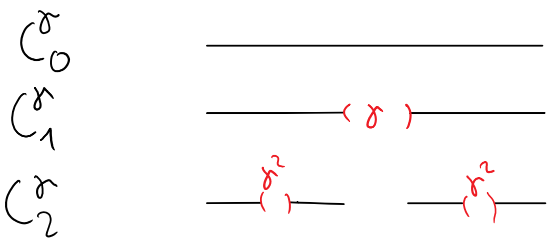
$$\frac{1}{2} = 3^{-s} \leq \sum_i \operatorname{diam}(B_i)^s.$$

Since $\{B_i\}$ was arbitrary this implies that $\mathcal{H}^s(C) \geq \frac{1}{2}$ and $\dim C \geq s$.

Note that the topological dimension of the Cantor set is 0 and in particular the Hausdorff dimension is strictly bigger than the topological dimension. Sutch objects are called fractal.

Instead of looking at the middle third Cantor set one can have a look at the generalized Cantor set $C_\gamma$, where $C^\gamma$ is derived by removing the middel interval of length $\gamma^k$ in the $k$-th iteration. The Hausdorff dimension of $C^\gamma$ is given by

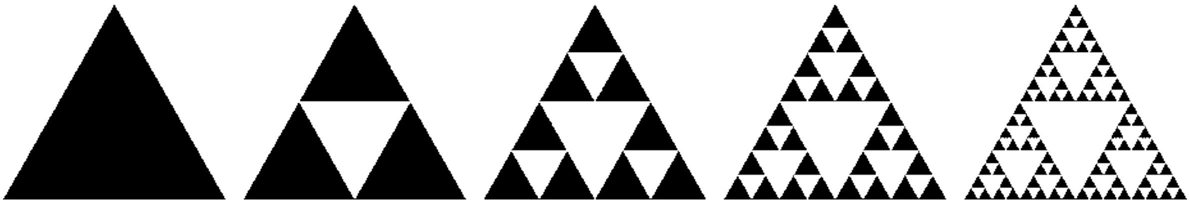
$$\dim C^\gamma = \frac{-\log 2}{\log\left(\frac{1-\gamma}{2}\right)}.$$




For $\gamma = \frac{1}{3}$ we get the statement for the middle third cantor set.
Another example is the Sierpinski triangle:

1. Start with an equiliteral triangle with a base parallel to the horizontal axes

2. Shrink triangle to 1/2 height and 1/2 width, make three copies, and position the three shrunken triangles at a corner. Note the emergence of the central hole because the three shrunken triangles can cover only 3/4 of the area of the original.
3. Repeat step 2 with each of the smaller triangles.



The Sierpinski triangle has Hausdorff dimension $\frac{\log 3}{\log 2}$ and topological dimension 1.

## References


1. V. Beresnevich, Metric Number Theory, Lecture notes University of York 2014
2. V. Beresnevich, F. A. Ramirez, S. Velani, Metric Diophantine Approximation: aspects of recent work, Dynamics and Analytic Number Theory, Cambridge University Press 2016.
3. Wikipedia, Sierpinski triangle, 25.03.2020.
4. Wikipedia, List of fractals by Hausdorff dimension, 25.03.2020.