Quadratic forms, Markov numbers and Diophantine approximation

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1. Continued fractions: basic facts. (**3rd year**)

   Continued fraction algorithm.

   Examples.

   Convergence properties of continued fractions.

   Bibliography: [4, sections 3.4-3.5]

2. Continued fractions, continuation. (**3rd year**)

   (Convergence properties of continued fractions, continuation.)

   More examples.

   Equivalent numbers. Serret's Theorem.

   Bibliography: [4, section 3.5], [9, section 10.11], [3], [5]

3. Approximation of irrationals by rationals. (**3rd year**)

   Convergents as best approximations.

   Proof of Hurwitz's Theorem.

   Lagrange spectrum: definition and structure.

   Bibliography: [5], [10], [9]

4. Badly approximable numbers. (**3rd year**)

   Definition in terms of rate of approximation and in terms of Lagrange spectrum.

   Characterization in terms of bounded partial quotients.

   Examples.

   Proof of Euler-Lagrange's Theorem on quadratic irrationalities.

   Bibliography: [5], [10]

5. Size of the set of badly approximable numbers.

   Proof of zero Lebesgue measure.

   Definition and basic properties of Haudorff measures and Hausdorff dimension.

   Example of the middle third Cantor set.

   Statement of Jarnik's Theorem.

   Bibliography: [5], [6], [10]

6. Schmidt's game applied to the set of badly approximable numbers

    Definition of the game and important implications of being 'winning'.

    Proof that the set of badly approximable numbers is winning.

    Bibliography: [15], [14], [2]

7. Markov's Theorem on quadratic forms

    Introduction to Markov's problem on minima of quadratic forms.

    Markov's Diophantine equation and Markov numbers.

    Markov's tree.

    Definition and properties of Markov's forms. Definition of Markov's quadratics.

    Proof of Markov's theorem for minima of quadratic forms.

    Bibliography: [8, Chap.II, sections 1-3-4-5], [1]

8. Markov's Theorem in Diophantine approximation

    Introduction to Markov's problem in Diophantine approximation.

    Markov's tree in terms of continued fraction expansions. The conjunction operation.

    Proof of Markov's Theorem for approximations.

    Bibliography: [8, Chap.II, sections 1-6], [16], [1], [7].

9. Hyperbolic geometry on Diophantine approximation

    The hyperbolic plane.

    Horocycles and geodesics.

    Ford circles and Farey tessellation.

    Proof of Hurwitz's theorem.

    Bibliography: [17, sections 4-5-6-7-8-9]

10. Markov's Problem from a Hyperbolic Geometry point of view

    Dictionary: Hyperbolic Geometry/Quadratic Forms.

    Geometric interpretation of Markov's Diophantine equation.

    Optimization problem on geodesics crossing a decorated ideal triangle.

    Bibliography: [17, sections 1-10-11-12-13]

11. Proof of Markov's Theorem using Hyperbolic Geometry

    Topological facts about simple closed geodesics and ideal arcs.

    Proof of Markov's Theorem.

    Bibliography: [17, sections 14-15].

12. Unicity/Uniqueness Conjecture

    Statement of the conjecture in both its original and the hyperbolic geometry form.

    Review of the partial results obtained up to now.

    Proof of the conjecture for Markov numbers that are prime powers.

    Bibliography: [11], [12], [17].

References

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