D-MATH
FS 2020
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Exercise Sheet 1

Algebraic Geometry

(1) Determine the radical of the ideal $(X_1^3 - X_2^6, X_1X_2 - X_2^3) \subseteq \mathbb{C}[X_1, X_2]$.
Hint: look at the vanishing locus.

(2) Let X be an affine variety. Show that the coordinate ring $A(X)$ is a field if and only if X is a single point.

(3) Let $X \subset \mathbb{A}^n$ be an affine variety. In class we defined the coordinate ring $A(X)$ to be the quotient ring $K[X_1, \ldots, X_n]/I(X)$. Alternatively, one can take the coordinate ring to be the set of all functions $f : X \to K$ such that there exists a polynomial $p \in K[X_1, \ldots, X_n]$ so that $f(c_1, ..., c_n) = p(c_1, ..., c_n)$. This set is a ring, with the obvious addition and multiplication of functions.

Show that the ring from the alternate definition is isomorphic to $A(X)$.

(4) Let $X \subseteq \mathbb{A}^n$ be an arbitrary subset. Prove that $V(I(X)) = \overline{X}$.

(5) Find the irreducible components of the affine variety $V(X_1 - X_2X_3, X_1X_3 - X_2^2) \subseteq \mathbb{A}^3_{\mathbb{C}}$.

(6) Let $f : X \to Y$ be a continuous map of topological spaces. Prove:

a. If X is irreducible then so is $f(X)$.

b. If X is connected then so is $f(X)$.

(7) Let A be an arbitrary subset of a topological space X. Prove that $\dim A \leq \dim X$.

(8) Let X be the set of all 2×3 matrices over a field K that have rank at most 1, considered as a subset of $\mathbb{A}^6 = \mathrm{Mat}(2 \times 3, K)$.
Show that X is an affine variety.

(9) Show that the ideal $I = (X_1X_2, X_1X_3, X_2X_3) \subseteq \mathbb{C}[X_1, X_2, X_3]$ cannot be generated by fewer than three elements. What is the zero locus of I?