D-MATH
FS 2020
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Exercise Sheet 2

Algebraic Geometry

1 Let $X = X_1 \cup \cdots \cup X_r$ be the decomposition of a Noetherian topological space into irreducible closed subsets with $X_i \not\subseteq X_j$ for all $i \neq j$. Show that

$$\dim X = \max\{\dim X_1, \ldots, \dim X_r\}.$$

2 Show that every Noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.

3 Let $\phi, \psi \in \mathcal{F}(U)$ be two sections of a sheaf $\mathcal{F}$ on an open subset U of a topological space X. Show:

a. If ϕ and ψ agree in all stalks, i.e. $\overline{(U,\phi)} = \overline{(U,\psi)} \in \mathcal{F}_a$ for all $a \in U$, then $\phi = \psi$.

b. If $\mathcal{F} = \mathcal{O}_X$ is the sheaf of regular functions on an irreducible affine variety X then we can already conclude that $\phi = \psi$ if we only know that they agree in one stalk $\mathcal{F}_a$ for $a \in U$.

c. For a general sheaf $\mathcal{F}$ on a topological space X the statement of b. is false.

4 Let a be any point on the real line $\mathbb{R}$. For which of the following sheaves $\mathcal{F}$ on $\mathbb{R}$ (with the standard topology) is the stalk $\mathcal{F}_a$ actually a local ring in the algebraic sense (i.e. it has exactly one maximal ideal)?

a. $\mathcal{F}$ is the sheaf of continuous functions;

b. $\mathcal{F}$ is the sheaf of locally polynomial functions.

5 Let Y be a non-empty irreducible subvariety of an affine variety X, and set $U = X - Y$. Assume that $A(X)$ is a unique factorization domain. Show that $\mathcal{O}_X(U) = A(X)$ if and only if $\operatorname{codim} Y \geq 2$.

6 Let X be a topological space. For each of the following, is it presheaf? Is it a sheaf in general?

a. $\mathcal{F}(U)$=continuous functions from U to $\mathbb{R}$.

b. $\mathcal{F}(U)$=bounded functions from U to $\mathbb{R}$.

c. $\mathcal{F}(U) = K$, where K is a field and the restriction maps are the identity.

d. Fix a point $x \in X$,

$$\mathcal{F}(U) = \begin{cases} K & \text{if} x \in U \\ 0 & \text{if} x \notin U. \end{cases}$$

(What should the restriction maps be?)

e. $\mathcal{F}(U) = U$.