D-MATH
FS 2020
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Exercise Sheet 3

Algebraic Geometry

1 Which of the following ringed spaces are isomorphic over $\mathbb{C}$?

a. $\mathbb{A}^1$;

b. $V(X_1^2 + X_2^2) \subseteq \mathbb{A}^2$;

c. $V(X_2 - X_1^2, X_3 - X_1^3) - \{0\} \subseteq \mathbb{A}^3$;

d. $V(X_1 X_2) \subseteq \mathbb{A}^2$;

e. $\mathbb{A}^1 - \{0\}$.

2 Let $Y = V(Y - Z^2, XZ - Y^2, YZ - X) \subseteq \mathbb{A}^3$. Define a map

$$\begin{aligned} f : \mathbb{A}^1 &\longrightarrow Y \\ t &\longmapsto (t^3, t^2, t). \end{aligned}$$

Check that this is a well defined morphism. (This is called the twisted cubic curve).

3 Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the corresponding homomorphism of the coordinate rings. Are the following statements true or false?

a. f is surjective if and only if f^* is injective.

b. f is injective if and only if f^* is surjective.

Hint: See examples 4.9 and 4.18 in the text (that we did in class).

c. If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism then f is affine linear, i.e. of the form $f(x) = ax + b$ for some $a, b \in K$.

d. If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism then f is affine linear, i.e. of the form $f(x) = Ax + b$ for some $A \in \mathrm{Mat}(2 \times 2, K)$ and $b \in K^2$.

4 Prove the following statements:

a. Every morphism $\mathbb{A}^1 - \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^1 \to \mathbb{P}^1$.

Hint: Let X_1, X_2 be the standard open cover of $\mathbb{P}^1$, as discussed in class. A morphism $\mathbb{A}^1 - \{0\} \to \mathbb{P}^1$ is the same as a pair of morphisms $f_i : \mathbb{A}^1 - \{0\} \to X_i$ that are compatible with the gluing maps.

b. Not every morphism $\mathbb{A}^2 - \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\mathbb{A}^2 \to \mathbb{P}^1$.

c. Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

(5) If X and Y are affine varieties we have seen that there is a bijection

$$\{\text{morphisms } X \to Y\} \overset{1:1}{\longleftrightarrow} \{K-\text{algebra morphisms } \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)\}$$
$$f \longmapsto f^*.$$

Does this statement still hold

a. if X is an arbitrary prevariety (but Y is still affine);

b. if Y is an arbitrary prevariety (but X is still affine)?
Hint: We see that $A(\mathbb{P}^1) = K$ (the only global functions are constant).