D-MATH
FS 2020
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Exercise Sheet 4

Algebraic Geometry

(1) Let Y be a closed subset of a prevariety X, considered as a ringed space with the structure sheaf making it a closed subprevariety. Prove for every affine open subset $U \subseteq X$ that the ringed space $U \cap Y$ (considered as an open subset of the ringed space Y) is isomorphic to the affine variety $U \cap Y$ (considered as an affine subvariety of the affine variety U).
In particular, this shows that a closed subprevariety Y is a prevariety, and it is isomorphic to the affine variety Y if X is itself affine (and thus Y an affine subvariety of X).

(2) Show that the space $\mathbb{P}^1$ of Example 5.5 (a) of Gathmann's notes is a variety.

(3) Use diagonals to prove the following statements:

a. The intersection of any two affine open subsets of a variety is again an affine open subset.
b. If $X, Y \subseteq \mathbb{A}^n$ are two pure-dimensional affine varieties then every irreducible component of $X \cap Y$ has dimension at least $\dim X + \dim Y - n$.

Hint: Use the fact from commutative algebra that the sum of the dimension and the codimension of a subvariety is the dimension of the ambient variety, and start by showing that $\operatorname{codim}(C(X)) \geq \operatorname{codim}(X) = n - \dim X$.

c. Now let $X \subseteq \mathbb{P}^n$ be a projective variety. Prove that the dimension of the cone $C(X) \subseteq \mathbb{A}^{n+1}$ is $\dim X + 1$.
d. Let $X, Y \subseteq \mathbb{P}^n$ be projective varieties with $\dim X + \dim Y \geq n$. Show that $X \cap Y \neq \emptyset$.

Hint: Recall, from commutative algebra, that $\dim(X \times Y) = \dim X + \dim Y$ for any two affine varieties X and Y.